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Academician Igor' Evgen'evich Tamm

(On the occasion of his 60th birthday)

J. Exper. Theoret. Phys. USSR **29**, 3-5 (July, 1955)

THE outstanding Soviet theoretical physicist, Igor' Evgen'evich Tamm, whose 60th birthday was celebrated last July 8, is among the number of scientists who played an exceedingly important role in the development of Soviet science.

Igor' Evgen'evich was born in 1895 in Vladivostok. From 1899 through 1913 he lived in Elizavetgrad (now Kirovgrad), where he was graduated from the gymnasium. He studied one year at the University of Edinburgh, but at the start of the First World War he transferred to the Moscow University, from which he was graduated in 1918 (Mathematics-Physics Department). Subsequently he carried on educational work in universities.

Igor' Evgen'evich has been working in Moscow since 1922. In 1925 L. I. Mandel'shtam invited him to the Moscow University, where he later headed the Theoretical Physics Department for many years. After the removal of the USSR Academy of Sciences to Moscow, Igor' Evgen'evich has been continuously in charge of the theoretical division of the P. N. Lebedev Institute of the Academy since 1934.

I. E. Tamm's versatile scientific creativeness covered many problems in quantum theory and its applications (to optics, physics of metals, cosmic-ray physics, radiation of fast particles, etc.), the problem of nuclear forces, and generally the principle problems of the theory of interaction of elementary particles, important application problems in theoretical physics, and many others.

In the early years of his scientific activity, I. E. Tamm published material on the electrodynamics of anisotropic media and on crystal optics, as well as on problems that led directly to the foundation of wave mechanics (correspondence principle, para-magnetism, etc.). Igor' Evgen'evich next applied himself actively to the development of the most pressing problems of quantum mechanics. Among them is the investigation (1930) of scattering of light in crystals, including the theory of combination scattering in crystals. These are outstanding for containing the first quantization of elastic waves in a solid body and the first introduction of the concept of elastic quanta (phonons), which was later to play such an important role in solid-state physics. Also dating to the same period are important investigations (1930) in Dirac's theory which at that time was in a very complicated state. Igor' Evgen'evich first systematically investigated quantum-mechanically the scattering of light by a free electron and confirmed the correctness of the Kline-Nishina equation derived earlier by using the correspondence principle,

In the same investigation he reached the fundamentally important conclusion that the Dirac equation for scattering of light of limitingly small frequency can be reconciled with Thomson's classic equation (this correspondence being of course essential) only if electron levels having a negative energy are taken into account. He also proved the inadmissibility of excluding from the theory these levels, which were so difficult to visualize at that time. It was here that Igor' Evgen'evich calculated the annihilation cross section of an electron with a "hole" in the background. For these calculations he employed a convenient method, which was later to receive wide application under the name of the Casimir method.

Also dating back to the early 30's is the research done by I. E. Tamm and his students in the quantum theory of metals. Outstanding among these are two important investigations. First was the investigation (1931-1933, together with S. P.

Shubin) that laid the foundation for the theory of the photo-effect in metals. Second was the investigation (1933) in which he predicted theoretically the existence of a special bound state of an electron on the surfaces of crystals. These "Tamm levels" later played a very important role in the theory of a great variety of surface effects in metals and semi-conductors, and now again play a part in the study of contact properties.

At the same time Igor' Evgen'evich started his research in nuclear theory.

In 1934 I. E. Tamm (together with S. A. Al'tshuler) analyzed experimental data obtained jointly with experimenters Becher and Shiuler and concluded that the neutron has a negative magnetic moment.

In 1934 I. E. Tamm published one of his most important works in which he derived the first equation for the potential of nuclear forces, based on a concrete model, and founded on the hypothesis that the nuclear forces are due to the interchange of electron-neutrino pairs. He also noted in the same communication that these "beta forces" are too weak to account for the observed nuclear interactions. As is well known, the situation was cleared up only a few years later, when mesons were discovered and when it became clear that it was the mesons that carried the nuclear forces. But even later the structure of the theory of nuclear forces was based on Tamm's work.

The study of nuclear and of elementary particles remains central in I. E. Tamm's scientific creativity for many years. However, along with this he carried out many other important investigations. In 1937 Igor' Evgen'evich (together with I. M. Frank) explained and devised a theory for the interesting phenomenon discovered by P. A. Cerenkov in S. I. Vasilov's laboratory; namely, the emission of radiation by a charged particle moving uniformly in a medium at a velocity exceeding the phase velocity of light.

In 1939 I. E. Tamm (together with S. Z. Belen'ko) developed a theory of the electron-photon shower, correctly accounting for the ionization losses of the particles. This was the first to disclose a possibility of accurate quantitative study of the process of cascade multiplication.

During the years of World War II and later, Igor' Evgen'evich performed many extremely valuable investigations of applied character. Among these we can mention, for example, the work on ionosphere currents, on electromagnetic processes in a laminated core, etc.

In 1945 I. E. Tamm returned to the interaction of elementary particles. He investigated the problem of stability of a deuteron, using a method which is now known as the "Tamm method". This method is based on the concept suggested in the theory of elementary particles by V. A. Fok in1933 ("the Fok functional method"). More than 100 scientific investigations have already been performed over the entire world with the "Tamm method".

In the same year I. E. Tamm published the work "On the Uncertainty Relationships for Time and Energy", written jointly with L. I. Mandel'shtam. This work is essential for the understanding of the principal problems in quantum mechanics. Of considerable interest is an investigation (performed together with V. L. Ginzburg) on the possibility of the existence of particles, which can be in a state with different spins.

We cannot even list all of I. E. Tamm's numerous works, covering a great variety of subjects. At the present time he carries on, together with a staff of theoreticians under his guidance, exceedingly active research principally in the region of the central problem of modern physics, namely, the theory of interaction of elementary particles. Here Igor' Evgen'evich is developing simultaneously two methods: a semi-phenomenological method, taking into account the possibility of "Isobar" state of the nucleons, and a new improved form of the "Tamm method".

Together with his research, I. E. Tamm continues to carry out considerable pedagogical and scientific-organizational work. For many years he has been in charge, and actually newly created the department of theoretical physics of the physical faculty of the Moscow University and of the Moscow Engineering-Physics Institute. He wrote one of the best university texts in the world, "Fundamentals of the Theory of Electricity", several editions of which were published. Among his students are more than ten doctors of science (including an academician and an associate member of the academy) and many candidates for science degrees.

I. E. Tamm's scientific activity is well recognized. He was elected an associate member of the Academy of Science of the USSR in 1933, and a full member in 1953. He was awarded the Stalin prize of first degree; he was given two Orders of Lenin, the Order of the Labor Red Banner and many other state awards.

I. E. Tamm enters his 60th birthday performing intense and fruitful work, in a period of new uplift of his creative work and in an atmosphere of recognition of his services and of general respect. The editor of the Journal of Experimental and Theoretical Physics extends to Igor' Evgen'evich warm wishes for health and new successes for the good of Soviet science.

Translated by J. G. Adashko

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Method of Terminated Field Equations and Its Applications to Scattering of Mesons by Nucleons

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Investigations of several general problems involved in the use of a new method of terminated equations are reported. This method is applied to the problem of scattering of mesons by nucleons with a higher degree of approximation than formerly.

1. INTRODUCTION

IN spite of the known shortcomings of modern quantum field theory, all the derivations of quantum electrodynamics that are capable of experimental verification have been fully corroborated experimentally. However, the same cannot be said of meson theory, principally because in meson theory, unlike in electrodynamics, it is impossible to expand the interaction constant in powers. Nevertheless, this does not exclude the possibility that modern meson theory agrees with experience even though in a certain limited energy region only. To resolve this problem it is necessary to solve the field equations by a method other than the conventional perturbation theory. Among the several such methods, the one that is being most intensely developed in recent times is the method of terminating the field equations in accordance with the number of virtual particles (we shall call this for brevity the method of terminated equations).

Its advantages over the so-called four-dimensional methods (for example, the Bethe--Salpeter method[1]) are first that the physical sense of all quantities employed in this method is quite clear, and second, that it permits obtaining directly from the system of four-dimensional covariant equations a relatively easily solvable system of equations in three-dimensional momentum space.

However, the value of this method will, in the final analysis, be determined by whether it is enough, when solving physical problems, even in a limited energy region, to take into account only a relatively small number of virtual particles in the sense that consideration of a larger number of particles will not affect substantially the result of calculations, and that this result will agree with experiment. So far it has not been possible to find a method for the general solution of this problem analytically, so that an answer to this question can apparently be obtained only by direct computation of various effects.

In our opinion, the principal hope for the possible success of this method is based on the fact that its application to the scattering of mesons by nucleons has led to a considerable theoretical success in explaining the experimentally observed resonance of a pi-meson + nucleon system in a $I = J = 3/2$ state at an energy on the order of 180 mev[2].

The method of terminated equations for meson dynamics was independently suggested by Tamm[3] in 1945 and Dancoff[4] in 1950; the latter, however, did not know that an analogous method was employed by Fok[5] in electrodynamics as early as 1934.

Without dwelling on the history of the development of the method, let us note only two of the most important recent investigations that contributed substantially to its perfection. Cini[6] proposed a covariant formulation of the method and first investigated the question of renormalization in this method. Dyson[7] suggested that the actual physical system be described by parameters that characterize the difference between this system and a physical rather than a mathematical vacuum; this enabled him particularly to eliminate from consideration vacuum loops, the treatment of which involves specific difficulties in the old method.

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[2] S. Fubini, Nuovo Cimento 10, 564 (1953); M. J. Dyson, M. Ross, E. E. Salpeter, S. S. Schweber, M. K. Sundaresan, W. M. Wirschera and H. A. Bethe, Phys. Rev. 95, 1644 (1954)

[3] I. E. Tamm, J. Phys. USSR 9, 445 (1945)

[4] S. M. Dancoff, Phys. Rev. 78, 382 (1950)

[5] V. A. Fok, Phys. Z. Sowjetunion 6, 425 (1934)

[6] M. Cini, Nuovo Cimento 10, 526, 624 (1953)

[7] F. Dyson, Phys. Rev. 90, 994; 91, 421, 1543 (1953)

We shall call the variant of the method as suggested by Dyson the "new method" of terminated equations.

This article first considers several general problems, arising when the "new method" is used, and then applies this method to meson scattering by nucleons, employing a much closer approximation than used in the past. Section 2 contains a formulation of the general rules for obtaining the system of covariant equations by the new method of terminated equations. The essential difference is emphasized between the above formulation of the method and the so called Levy-Klein method[8], which, generally speaking, results in a divergent series in powers of the coupling constant for the unknown amplitude in the kernel. Section 3 discusses the transformation of the covariant equations into the three-dimensional momentum space. Section 4 is devoted principally to the selection of the correct boundary equations when solving various problems by the new method of terminated equations. It is shown that the boundary conditions can be formulated in the new method by using the connection between the amplitudes obtained by the old and new methods.

In Sec. 5, the new method is used to investigate the scattering of pi-mesons by nucleons. An approximation is discussed in which the contribution of the so-called "minus-particles" to the amplitude of the scattering is taken into account. Section 6 contains a report on the basic results of the analysis of the self-energy terms arising in the equations for the meson + nucleon system.

It is quite difficult to eliminate successively the divergent terms in the equation for the meson + nucleon system in the approximation considered in this article. First af all, two renormalized charges[9] result from the renormalization of the self-energy kernel; second, calculation of the finite additions due to these kernels leads to additional poles in these equations (see also reference 10); finally, unlike the solution for the interaction of two nucleons, the solution for the meson + nucleon system with $I = 1/2$ contains singularities, which must be eliminated by additional renormalization[11]. These problems are not considered in this article.

We have carried out numerical calculations for the case $I = 3/2$ without taking into account the self-energy terms, but taking into account the "minus particles",

2. GENERAL FORMULATION OF THE "NEW" METHOD

We shall employ the interaction representation. Let $\Psi(\sigma)$ be the state vector of the system under consideration on a space-like three-dimensional surface σ; let $\Psi_0(\sigma)$ be the corresponding physical-vacuum vector.

Let $H(x)$ be the Hamiltonian of the interaction at the four-dimensional point x lying on the surface σ. The wave functions satisfy the well-known equations:

$$i\frac{\delta\Psi(\sigma)}{\delta\sigma(x)} = H(x)\Psi(\sigma),$$
$$-i\frac{\delta\Psi_0^*(\sigma)}{\delta\sigma(x)} = \Psi_0^*(\sigma)H(x). \tag{2,1}$$

Let $\psi_\alpha^\lambda(x)$ and $\overline{\psi}_\alpha^\lambda(x)$ be the nucleon-field operators in the interaction representation and let $\phi^s(x)$ be the meson-field operators. Here α is the spinor index ($\alpha = 1, 2, 3, 4$), λ the isotopic-spin index of the nucleon ($\lambda = 1, 2$), and s the isotopic-spin index of the meson ($s = 1, 2, 3$). We shall frequently drop these indices. Operators ψ, $\overline{\psi}$, and ϕ obey the usual commutation rules:

$$[\psi_\alpha^\lambda(x_1), \overline{\psi}_\beta^\mu(x_2)]_+ = -i\delta_{\lambda\mu}S_{\alpha\beta}(x_1 - x_2), \tag{2.2}$$

$$[\varphi^s(x_1), \varphi^{s'}(x_2)]_- = -i\delta_{ss'}\Delta(x_1 - x_2).$$

We shall term the quantities of the following type:

$$\langle\psi(x_1)\rangle_\sigma \equiv \Psi_0^*(\sigma)\psi(x_1)\Psi(\sigma), \tag{2.3}$$
$$\langle\psi(x_1)\varphi(x_2)\rangle_\sigma \equiv \Psi_0^*(\sigma)\psi(x_1)\varphi(x_2)\Psi(\sigma),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\langle\overline{\psi}(x_1)\ldots\overline{\psi}(x_n)\psi(y_1)\ldots\psi(y_m)\varphi(z_1)\ldots\varphi(z_r)\rangle_\sigma \equiv$$
$$\equiv \Psi_0^*(\sigma)N\overline{\psi}(x_1)\ldots\overline{\psi}(x_n)\psi(y_1)\ldots\psi(y_m)\varphi(z_1)\ldots\varphi(z_r)\Psi(\sigma),$$

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[10] V. M. Visscher, Phys. Rev. 96, 788 (1954)

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the "$n + m + r$ partial amplitudes". The four-dimensional points x_1, x_2, etc. are quite arbitrary and do not have to lie on the surface σ. In Eqs. (2.3), N denotes an ordered operator product, that is, each field operator ψ, $\overline{\psi}$, or ϕ is represented in the form of a sum of the particle absorption and creation operators, while all absorption operators lie to the right of the creation operators in the N product. The following very simple examples show the difference between the N-ordered product and the conventional one:

$$\overline{\psi}^{\lambda}_{\alpha}(x_1)\,\psi^{\mu}_{\beta}(x_2) = N\overline{\psi}^{\lambda}_{\alpha}(x_1)\,\psi^{\mu}_{\beta}(x_2)$$

$$- i\delta_{\lambda\mu}S^{(-)}_{\beta\alpha}(x_2 - x_1),$$

$$\varphi_s(x_1)\,\varphi_{s'}(x_2) = N\varphi_s(x_1)\,\varphi_{s'}(x_2)$$

$$- i\delta_{ss'}\Delta^{(+)}(x_1 - x_2),$$

where the ± indices denote the positive- and negative-frequency portions of the permutation functions S and Δ. To transform analogously more complicated operator products it is necessary to employ the well-known theorem by Wick[12].

It is easy to obtain for Eq. (2.1) an equation for the amplitudes

$$i\frac{\delta}{\delta\sigma(x)}\langle\psi(x_1)\rangle_\sigma = \langle[\psi(x_1),\ H(x)]\rangle_\sigma, \qquad (2.4)$$

and analogous equations for the multiple-particle amplitudes.

If the type of Hamiltonian is made more specific and if we assume

$$H(x) = ig\tau^{s}_{\lambda\mu}\overline{\psi}^{\lambda}(x)\,\gamma^5\,\psi^{\mu}(x)\,\varphi_s(x), \qquad (2.5)$$

where $\tau^{s}_{\lambda\mu}$ is the matrix of the isotopic spin of the nucleon, and $\gamma^5 = -i\gamma^1\ \gamma^2\gamma^3\gamma^4 = \rho_1$ (we use the Feynman form of the Dirac matrix γ^k), we can transform Eq. (2.4) with the aid of Eq. (2.2) into:

$$i\frac{\delta}{\delta\sigma(x)}\langle\psi_\lambda(x_1)\rangle_\sigma \qquad (2.6)$$

$$= g\tau^{s}_{\lambda\mu}S(x_1 - x)\,\gamma^5\langle\psi^{\mu}(x)\,\varphi_s(x)\rangle_\sigma.$$

Analogously, we can also obtain equations for the other amplitudes. In general, several amplitudes appear in the right half. The infinite system of equations that generally results is a rigorous covariant field-equation system. It can be subdivided into independent sub-systems in accordance with the magnitude of the nuclear charge – the number of operators ψ and the number of adjoint operators $\overline{\psi}$ in each of the sub-systems is constant.

To obtain an approximate "terminated" system of equations, it is necessary to discard from the rigorous system of equations all amplitudes for a number of particles exceeding a certain number n_0. As a result we obtain a complete, and as can be shown also a common "terminated" system of equations for determining the ordered amplitudes, with the number of particles not exceeding the number n_0.

When performing the calculations, it is convenient to select the three-dimensional surfaces σ such that t is constant. Integrating the differential equations for the amplitudes of the type (2.4) over the three-dimensional space, we obtain equations of the type.

$$i\frac{\partial}{\partial t}\langle\psi(x_1)\rangle_t = \int d\mathbf{r}\,\langle[\psi(x_1),\ H(x)]\rangle_t, \qquad (2.7)$$

where $x = (\mathbf{r}, t)$.

Many investigators[8,13] working with the so-called Tamm-Dancoff equations understand this term to mean one equation for one amplitude, corresponding to the actual particle system specified for the given problem (for example, nucleon + meson in scattering theory or two nucleons in deuteron theory), with this equation being obtained from the above-described system of coupled equations for n various amplitudes by approximate elimination of all amplitudes but one single one. In practice this elimination reduces to the fact that the kernel of the rigorous integral equation for the separated amplitude is expanded in powers of the interaction constant g and is terminated at a certain power of this constant. However, since it is known[8] that the expansion of the kernel in powers of g is generally divergent, such a simplification of the system of terminated equations may lead to incorrect results.

Let us note that approximately half of the amplitudes contained in these equations can always be eliminated. In fact, as follows from the above discussion, the derivatives of the amplitudes with odd number of particles are expressible in terms of amplitudes with even number of particles, and vice versa. Therefore, integration of equations of the type (2.7) with respect to time always permits expressing, say, all odd-particle amplitudes in terms of even-particle ones, thus completely eliminating the odd amplitudes from the system of equations.

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[12] G. C. Wick, Phys. Rev. 80, 268 (1950); A. I. Akhiezer and V. B. Berestetskii, Kvantovaia elektrodinamika (Quantum Electrodynamics), GITTI, Moscow, 1953

[13] M. Levy, Phys. Rev. 88, 72, 725 (1952); F. Macke, Z. Naturforsch 8a, 594 (1954); W. Zimmerman, Suppl. Nuovo Cimento 11, 43 (1954); J. C. Taylor, Phys. Rev. 95, 1313 (1954)

3. TRANSFORMATION TO MOMENTUM REPRESENTATION

Let us choose the surface σ such that t is constant. Let us restrict our discussion to the stationary states of the physical systems. The state vector of such states has the following form in the interaction representation:

$$\Psi(t) = \exp\{i(H_0 - \mathcal{E})t\}\,\Psi'. \tag{3.1}$$

The physical-vacuum vector has an analogous form

$$\Psi_0(t) = \exp\{i(H_0 - \mathcal{E}_0)t\}\,\Psi_0', \tag{3.2}$$

where H_0 is the Hamiltonian of the free, non-interacting nucleon and meson fields, $\mathcal{E}$ is the energy of the given system, $\mathcal{E}_0$ is the energy of the physical vacuum, and finally Ψ' and Ψ_0' are constant vectors in the functional space.

Let us expand operators ψ, $\overline{\psi}$, and ϕ in a three-dimensional Fourier series, using the known equations (3.3)

$$\varphi_s(x) = \frac{1}{L^{3/2}}\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(Q_s(\mathbf{k})\,e^{-ikx} + Q_s^*(\mathbf{k})\,e^{ikx}\right),$$

$$\psi_\alpha^\lambda(x) = \frac{1}{L^{3/2}}\sum_{n=1}^{4}\sum_{p} e^{-ipx}B_n^\lambda(\mathbf{p})\,u_\alpha^n(\mathbf{p}),$$

with corresponding expressions for $\overline{\psi} = \psi^*\gamma_4$. Here

$$kx = \omega_k t - \mathbf{kr},\quad \omega_k = \sqrt{\mu^2 + \mathbf{k}^2}, \tag{3.4}$$

Q_s and Q_s^* are respectively the absorption and emission operators for mesons of the s type, whereby

$$[Q_s(\mathbf{k}),\ Q_s^*(\mathbf{k}')]_- = \delta_{ss'}\delta_{\mathbf{kk}'}, \tag{3.5}$$

$u_\alpha^n(\mathbf{p})$ are four bi-spinors ($n = 1, 2, 3, 4$) satisfying the Dirac equation for the given $\mathbf{p}$ and satisfying the orthogonality conditions

$$\sum_\alpha u_\alpha^{n*}(\mathbf{p})\,u_\alpha^{n'}(\mathbf{p}) = \delta_{nn'}, \tag{3.6}$$

with the indices $n = 1, 2$ corresponding to solutions with positive energy, and $n = 3, 4$ corresponding to solutions with negative energy. Furthermore,

$$px = p_0 t - \mathbf{pr},\ p_0 = \delta_n E_p, \tag{3.7}$$

$$E_p = \sqrt{M^2 + \mathbf{p}^2},$$

where $\delta_n = +1$ for $n = 1, 2$ and $\delta_n = -1$ for $n = 3, 4$. Finally $B_1^\lambda(\mathbf{p})$ and $B_2^\lambda(\mathbf{p})$ are the absorption operators for nucleons of the λ type ($\lambda = 1$ for a proton and $\lambda = 2$ for a neutron) with momentum $\mathbf{p}$, while $B_3^\lambda(\mathbf{p})$ and $B_4^\lambda(\mathbf{p})$ are the emission operators for anti-nucleons with momentum $-\mathbf{p}$, whereby

$$[B_n^{\lambda*}(\mathbf{p}),\ B_{n'}^{\lambda'}(\mathbf{p}')]_+ = \delta_{\lambda\lambda'}\delta_{nn'}\delta_{\mathbf{pp}'}. \tag{3.8}$$

With the aid of Eqs. (3.1) and (3.3) it is possible to express the time-dependent amplitudes of the $\langle\psi(x_1)\rangle$ type introduced in Sec. 2 in terms of the time-independent amplitudes of the type

$$\langle B_n^\lambda(\mathbf{p})\rangle \equiv \Psi_0'^*B_n^\lambda(\mathbf{p})\,\Psi'. \tag{3.9}$$

Thus, for example, taking into account the usual commutation rule of the free-field Hamiltonian H_0 with the operators B_n^λ we obtain from Eqs. (2.7), (3.1) and (3.3)

$$\langle\psi(x_1)\rangle_t = \frac{1}{L^{3/2}}\sum_n\sum_{\mathbf{p}} u_\alpha^n(\mathbf{p}) \tag{3.10}$$

$$\exp\{-ipx_1 + i(p_0 - W)t\}\,\langle B_n^\lambda(\mathbf{p})\rangle,$$

where $W = \mathcal{E} - \mathcal{E}_0$ is the difference between the energy $\mathcal{E}$ of the system under consideration and the energy $\mathcal{E}_0$ of physical vacuum. Thus W is equal to the observed energy of the system.

To change over to momentum representation it is also necessary to employ the well-known expressions for the permutation functions:

$$\Delta(x) = \frac{i}{(2\pi)^3}\int d^4k\,e^{-ikx}\delta(k^2 - \mu^2)\,\varepsilon(k_0), \tag{3.11}$$

$$S(x) = \frac{i}{(2\pi)^3}\int d^4p\,e^{-ipx}\delta(p^2 - M^2)\,\varepsilon(p_0)\,(\hat{p} + M),$$

where $p = p_\mu\gamma^\mu$ and

$$\varepsilon(k_0) = 1 \quad \text{при} \quad k_0 > 0, \qquad (3.12)$$
$$\varepsilon(k_0) = -1 \quad \text{при} \quad k_0 < 0.$$

With the aid of Eqs. (3.1), (3.3) and (3.11), it is possible to transform the system of covariant terminated equations in four-dimensional coordinate space, described in Sec. 2, into a system of equations for stationary amplitudes of the $< B_n^\lambda(\mathbf{p}) >$ type in three-dimensional momentum space. Thus, for example, if this transformation is carried out for Eq. (2.7), we obtain two equations, one for the quantity $\sum_{n=1}^{2} u_\alpha^n(\mathbf{p}) \langle B_n^\lambda(\mathbf{p}) \rangle$ and the other for $\sum_{n=3}^{4} u_\alpha^n(\mathbf{p}) \langle B_n^\lambda(\mathbf{p}) \rangle$. Using the orthogonality of the functions $u_\alpha^n(\mathbf{p})$ [Eq. (3.6)] we can reduce both equations into the following form

$$(W - \delta_n E_p) \langle B_n^\lambda(\mathbf{p}) \rangle \qquad (3.13)$$
$$= \frac{ig}{2E_p} \tau_{\lambda_\mu}^s \delta_n u^{n^*}(\mathbf{p}) (M + \hat{\mathrm{p}}) \gamma^5$$
$$\times \int d\mathbf{q} \sum_{n'=1}^{4} u^{n'}(\mathbf{q}) \frac{1}{\sqrt{2\omega_{\mathbf{p}-\mathbf{q}}}}$$
$$\times \{\langle B_{n'}^\mu(\mathbf{q}) Q_s(\mathbf{p}-\mathbf{q}) \rangle + \langle B_{n'}^\mu(\mathbf{q}) Q_s^*(\mathbf{q}-\mathbf{p}) \rangle\},$$

where $\hat{\mathrm{p}} = \delta_n E_n \gamma_4 - \mathbf{p}\vec{\gamma}$.

Let us remark that general transformations of the equations, particularly the elimination of a portion of the amplitudes from the system of equations and their renormalization, is best carried out in the covariant equations written in the coordinate space. On the other hand, elimination of angles and numerical solutions to the equations must be performed in the momentum representation.

4. "MINUS PARTICLES". BOUNDARY CONDITIONS

Let us return to the question of the meaning of the stationary amplitudes of the $< B(\mathbf{p}) >$ type, which we shall call the Dyson amplitudes.

In the old method of terminated equations, the state of the system Ψ' was characterized by the aggregate of amplitudes of the following type:

$${}_0\langle B(\mathbf{p}) \ldots Q(\mathbf{k}) \ldots \rangle \qquad (4.1)$$
$$\equiv \Phi_0^{\prime *} B(\mathbf{p}) \ldots Q(\mathbf{k}) \ldots \Psi'.,$$

where Φ_0' is the state vector of mathematical vacuum. These amplitudes have the sense of amplitudes of the probability that the state Ψ' has a prescribed number of particles with definite momenta. From the definition of mathematical vacuum it follows that all ordered amplitudes of the type of Eq. (4.1), in which the particle-emission operators are encountered, equal zero.

The Dyson amplitudes (3.9) differ from the old amplitudes (4.1) in that the state vector Φ_0' of the mathematical vacuum is replaced in them by the state vector Ψ_0' of the physical vacuum. Therefore, the Dyson amplitudes, generally speaking, also differ from zero in that case when they contain particle-emission operators. Thus, for example, the amplitude $< Q^*(\mathbf{k}) > = \Psi_0^{*\prime} Q^*(\mathbf{k}) \Psi'$ describes the probability that the state Ψ' differs from physical vacuum by the fact that it is short one meson having a momentum $-\mathbf{k}$ contained in the state Ψ_0', i.e., in vacuum. In this case we shall say that the state Ψ' contains one "minus-meson" (or correspondingly, a "minus nucleon" or a "minus-antinucleon"). Let us note the important fact that if the Dyson amplitudes are found for a given state, it is possible to determine from them both the "old" amplitudes of the (4.1) type, of state Ψ', as well as the vacuum amplitudes *

$$\Phi_0^{\prime *} B(\mathbf{p}) \ldots Q(\mathbf{k}) \ldots \Psi_0. \qquad (4.2)$$

Thus the solution of any physical problem obtained by the new method is equivalent to the solution of the same problem by the old method and is simultaneously equivalent to the solution of the problem of determining the amplitudes (4.2) that characterize the state of physical vacuum.

The system of equations, satisfied by the Dyson amplitudes, has the following form:

$$(W - \delta_n E_p - \delta_{n'} E_{p'} - \cdots - \xi_k \omega_k) \qquad (4.3)$$
$$\langle B^n(\mathbf{p}) B^{n'}(\mathbf{p}') \ldots Q^{\xi_k}(\mathbf{k}) \rangle$$
$$= X^{nn' \ldots \xi_k}(\mathbf{p}, \mathbf{p}', \ldots, \mathbf{k}),$$

---

* See reference 7, Eq. (26), from which we can determine the ratio of any two "old" amplitudes for both states Ψ' and Ψ_0'.

in which we introduce the notation $\xi_k = \pm 1$

$$Q^{\xi_k}(\mathbf{k}) = Q(\mathbf{k}), \quad \text{if} \quad \xi_k = +1, \tag{4.4}$$
$$Q^{\xi_k}(\mathbf{k}) = Q^*(\mathbf{k}), \quad \text{if} \quad \xi_k = -1,$$

and where $X(\mathbf{p}, \mathbf{p}', \ldots, \mathbf{k})$ are definite linear functions of the Dyson amplitudes.

If the factor in the left half of Eq. (4.3) does not vanish, this equation has a unique solution

$$\langle B^n(\mathbf{p}) B^{n'}(\mathbf{p}') \ldots Q^{\xi_k}(\mathbf{k}) \rangle \tag{4.5}$$
$$= \frac{X^{nn' \ldots \xi_k}(\mathbf{p}, \mathbf{p}', \ldots, \mathbf{k})}{W - \delta_n E_p - \delta_{n'} E_{p'} - \ldots - \xi_k \omega_k}.$$

On the other hand, if the factor in the left half of Eq. (4.3) vanishes, then it is possible to add the quantity $\delta(W - \delta_n E_p - \delta_{n'} E_{p'} - \ldots - \xi_k \omega_k)$, multiplied by an arbitrary factor, to the right half of the solution (4.5), which for definiteness we shall always take in this case to be the principal value. To eliminate such indeterminacy it is necessary, as is customary, to take into account the boundary conditions, without which any physical problem is indeterminate. To formulate correct boundary conditions, we employ the connection between the Dyson amplitudes and the amplitudes of the old method. Namely, we employ Eq. (26) of reference 7:

$$a(N, N') \tag{D}$$
$$= \sum_M \beta^*(N + M)\, \alpha(N' + M)\, C(N, N'; M),$$

where a is the new-method amplitude, α is the old-method amplitude for the same state, β the vacuum-state amplitude, the C's are numerical coefficients, and N and N' denote respectively the number of plus and minus particles. Let us remark first of all that the vacuum amplitudes $\beta(N)$ should not contain delta-functions (if for no other reason than relativistic considerations of the invariance of the energy E_0 of the vacuum state after renormalization, which should be zero). This leads to the fact that in Eq. (D) delta-functions can occur only because of the amplitudes $\delta(N' + M)$ (since $E = W$ when $E_0 = 0$). In fact, in the case of the amplitudes $\alpha(N' + M)$ it is possible to obtain delta-functions in the form $\delta(W - E_{N'} - E_M)$. Consequently, when $N \neq 0$, $a(N, N')$ cannot contain $\delta(W + E_N - E_{N'})$, and when $N = 0$ it can contain $\delta(W - E_{N'})$. However, this appearance of delta-functions for amplitudes of states containing only plus particles is determined by the usual boundary conditions. On the other hand, if singularities of the type $(W + E_N - E_{N'})^{-1}$ occur for the amplitudes of states containing at least one minus particle , they must be considered in the sense of being principle values[7].

Let us make two remarks. First, when dealing with the stationary state, as we are doing at all times, we do not encounter any questions concerning the manner in which the interaction is "turned on". Second, in the method of terminated equations, the equation of a collision between a meson and a nucleon corresponds to the question of collisions in three-dimensional space between real particles (in the given approximation) rather than "bare" ones. To explain the latter, let us consider the amplitudes that are coupled in the system of terminated equations with the zero-approximation amplitudes

$$\langle B^n(\mathbf{p})\, Q(-\mathbf{p}) \rangle = \delta(\mathbf{p} - \mathbf{p}_0)\, \chi_n, \tag{4.6}$$

where $\chi_n = 0$ when $n = 3, 4$), describing the motion of the "bare" non-interacting nucleons and meson in terms of their center-of-inertia coordinates. These amplitudes describe not only the collision of the meson with the nucleon, but also the accumulation of a cloud of virtual particles about the "bare" nucleon and "bare" meson. It must be emphasized that such an accumulation takes place also for a meson and nucleon that are infinitely remote from each other (in three-dimensional space, corresponding to momentum space). Thus, for example, substitution of (4.6) into the equation for the amplitude $< B(\mathbf{p})\, Q(\mathbf{k})\, Q(\mathbf{l}) >$ discloses that the expression for this amplitude contains a term proportional to $\delta(\mathbf{l} + \mathbf{p}_0)\, \delta(\mathbf{p} + \mathbf{k} - \mathbf{p}_0)$ and describing a state in which the primary "bare" electron moves with a momentum $-\mathbf{p}_0$, and the primary nucleon is dissociated into a nucleon $\mathbf{p}$ and a meson $\mathbf{k}$ with a total momentum $\mathbf{p}_0$. What is significant is that here the amplitude $<B(\mathbf{p})\, Q(\mathbf{k})\, Q(\mathbf{l})>$ is expressed in terms of the amplitude $<B(\mathbf{p}) \times Q(-\mathbf{p})>$ by an equation similar to Eq. (4.5), containing no delta-function whatever in its right half.

Having established the boundary conditions for the equations in momentum space, it is possible to formulate the corresponding conditions for the covariant equation in coordinate space. As we have seen in Sec. 3, equations of the (2.6) type transformed into momentum space have the following form:

$$i\frac{\partial}{\partial t}A(x_1, x_2, \ldots; t) = \int d\mathbf{r}\, Y(x_1, x_2, \ldots; \mathbf{r}, t) = \int d\mathbf{r}\int d\nu f(x_1, x_2, \ldots, \mathbf{r}, \nu)\, e^{-i\nu t}, \tag{4.7}$$

where A denotes any amplitude of the (2.6) type. The solution of this equation is

$$A(x_1, x_2, \ldots, t) = \int d\mathbf{r}\int \frac{d\nu}{\nu} f(x_1, x_2, \ldots \mathbf{r}, \nu)\, e^{-i\nu t} + a(x_1, x_2, \ldots), \tag{4.8}$$

where $a(x_1, x_2, \ldots)$ is an arbitrary time-independent function, and the integral with respect to ν must be understood in the principal-value sense (ν vanishes within the integration region). It is convenient to rewrite Eq. (4.8) as follows:

$$A(x_1, x_2, \ldots; t) = {}^1\!/_2\int dx'\varepsilon(t-t')\, Y(x_1, x_2, \ldots; x') + a(x_1, x_2, \ldots), \tag{4.9}$$

where the integration is performed over the entire four-dimensional space. However, such a representation must be understood to imply the condition that the value of the integral for $t = \pm\infty$ must be dropped from the resultant expression. The function $a(x_1, x_2, \ldots)$ in momentum space corresponds to a delta-function describing plane and spherical waves with an energy W. From the boundary conditions formulated above for the momentum space, it follows that for all states in which there is at least one minus particle the function $a(x_1, x_2, \ldots)$ must be set equal to zero. For states containing only plus particles, this function must also be set equal to zero if the freqeuncy ν does not vanish within the integration region, otherwise the function a does vanish and should not be so selected that the function A corresponds to plane and outgoing plus-particle waves with a total energy W, corresponding to the conditions of the problem.

In particular, in the scattering of a meson by a nucleon at energies W that are insufficient for the formation of a second free meson, it follows from the above that the function vanishes for all amplitudes except for the state amplitude $< B(\mathbf{p})Q(-\mathbf{p}.)$ (one "bare" meson and one "bare" nucleon).

5. EQUATIONS FOR THE MESON + NUCLEON SYSTEM

Let us restrict ourselves to consideration of the states of a system having not more than three virtual particles. Accordingly, we write down equations similar to Eq. (2.6) for the principal amplitude of the problem $<\psi(x_1)\times\phi(x_2)>_t$ and for the amplitudes that couple with it

$$\langle\psi(x_1)\rangle_t, \quad \langle\psi(x_1)\varphi(x_2)\varphi(x_3)\rangle_t$$

and

$$\langle\bar{\psi}(x_1)\psi(x_2)\psi(x_3)\rangle_t.$$

(nucleon, nucleon + two mesons and nucleon + pair). The right halves of the equations for the latter two amplitudes contain the principal amplitude $<\psi(x_1)\phi(x_2)>_t$ and the four-dimensional amplitudes $<\bar{\psi}(x_1)\psi(x_2)\psi(x_3)\phi(x_4)_t$ and $<\psi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)>_t$; the latter will be omitted in our approximation.

Integrating the differential equations for the single-particle and three-particle amplitudes, we express them in accordance with Eq. (4.9)* in terms of their right halves, i.e., in terms of the amplitude $<\psi(x_1)\phi(x_2)>_t$. The resultant expressions for the single-particle and three-particle amplitudes are inserted into the right half of the equation for the amplitude $<\chi(x_1)\phi(x_2)>_t$ to obtain the final equation for this amplitude:

$$i\frac{\partial}{\partial t}\langle\psi^\lambda(x_1)\,\varphi_s(x_2)\rangle_t = \frac{1}{2}g^2\int d\mathbf{r}dx'\varepsilon(t-t')\{N_1Y_1 + N_2Y_2 + Y_n + Y_m\}, \tag{5.1}$$

where the following notation is used: (5.2)

$$Y_1 = S(x_1 - x)\gamma^5(ZS(x - x')\Delta(x' - x_2))\gamma^5 \times \langle\psi^\mu(x')\varphi_s(x)\rangle_{t'}$$

---

* Following the argument in Sec. 4, it is unnecessary to introduce into this equation additional terms of the type $a(x_1, \ldots)$ if the energy of the system is insufficient for emission of a second free meson.

$$-\Delta(x-x_2)(ZS(x_1-x')\gamma^5 S(x'-x))\gamma^5 \langle\psi^{\mu}(x)\varphi_{s'}(x')\rangle_{I'};$$

$$Y_2=(ZS(x_1-x)\Delta(x-x_2))\gamma^5 S(x-x')\gamma^5 \langle\psi^{\mu}(x')\varphi_{s'}(x')\rangle_{I'};$$

$$Y_n=3S(x_1-x)\gamma^5(ZS(x-x')\Delta(x'-x))\gamma^5 \langle\psi^{\lambda}(x')\varphi_s(x_2)\rangle_{I'};$$

$$Y_m=2\Delta(x-x_2)\,\mathrm{Tr}\,(ZS(x-x')\gamma^5 S(x'-x)\gamma^5) \langle\psi^{\lambda}(x_1)\varphi_s(x')\rangle_{I'}.$$

The symbol Z has the following meaning:

$$(ZS(x')\Delta(x''))=S^{(+)}(x')\Delta^{(-)}(x'') - S^{(-)}(x')\Delta^{(+)}(x'') \tag{5.3}$$

with analogous consideration when $\Delta(x'')$ is replaced by $S(x'')$; the symbol Tr denotes the trace of the matrix relative to the spin indices. Finally, N_1 and N_2 are operators in the isotopic-spin space

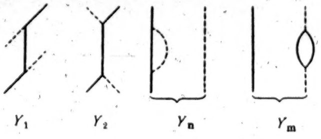
$$N_1=\tau^{s'}_{\lambda\nu}\tau^{s}_{\nu\mu},\quad N_2=\tau^{s}_{\lambda\nu}\tau^{s'}_{\nu\mu}. \tag{5.4}$$

The eigenvalues of these operators are:

$$\begin{aligned} &N_1=-1,\ N_2=3, \quad \text{if}\quad I={}^1/_2,\\ &N_1=2,\quad N_2=0, \quad \text{if}\quad I={}^3/_2, \end{aligned} \tag{5.5}$$

where I denotes the total isotopic spin of the meson + nucleon system. From now on we shall no longer write down the isotopic indices λ and s, and will take the operators N_1 and N_2 to mean their eigenvalues (5.5).

Diagrams representing the kernel Y_i of Eq. (5.1) are given in the Figure.



The first two kernels correspond to the scattering of a meson by a nucleon, whereby kernel Y_1 corresponds to scattering with the initial emission of a meson ("chain with emission") and Y_2 corresponds to the initial absorption of the meson ("chain with absorption"). These kernels are finite while the kernels Y_n and Y_m corresponding to the self energy of the nucleon and meson, respectively, are infinite and must be renormalized (as incidentally must be the finite kernel Y_2, see below). The renormalization of the self-energy terms will be considered separately; in this section we shall not consider these terms at all and will examine only the finite kernels Y_1 and Y_2. Let us note that Eq. (5.1) takes into account all the processes corresponding to all possible iterations of the diagrams in the Figure.

The transition from Eq. (5.1) to the momentum representation is carried out quite analogously with that performed in Sec. 3. We shall use a system of coordinates with the origin ot the center of inertia of the meson and nucleon. Let $\mathbf{p}_0$ be the momentum of the falling nucleon, and $-\mathbf{p}_0$ that of the falling meson, so that the energy of the system is

$$W=E_0+\omega_0=\sqrt{M^2+\mathbf{p}_0^2}+\sqrt{\mu^2+\mathbf{p}_0^2}. \tag{5.6}$$

It is convenient to introduce the following designations for the amplitudes

$$b^{\varepsilon\xi}_{\alpha}(\mathbf{p})=\langle B^{n}(\mathbf{p})\,Q^{\xi p}(-\mathbf{p})\rangle, \tag{5.7}$$

where ϵ, like ξ, assumes the values ± 1 ($\epsilon=+1$ for $n=1, 2$, i.e., for plus nucleons and $\epsilon=-1$ for $n=3, 4$, i.e., for minus antinucleons); the index $\alpha=1, 2$ differentiates among the possible (mechanical) spin directions of the nucleons and antinucleons. Let us also introduce the Pauli matrices σ acting on the spin index α, and finally also the designation

$$a^{\varepsilon\xi}(\mathbf{p})=\left(\frac{\vec{\sigma}\mathbf{p}}{p}\right)^{(1-\varepsilon)/2} b^{\varepsilon\xi}(\mathbf{p}) \tag{5.8}$$

(we omit the spin indices). Using these designations, Eq. (5.1) (without the Y_n and Y_m terms) assumes the following form in the momentum representation:

$$(W-\varepsilon E-\xi\omega)\,a^{\varepsilon\xi}(\mathbf{p}) = \frac{\lambda}{4\pi}\sum_{\varepsilon'\xi'}\int d\mathbf{p}'\,R^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\ \mathbf{p}')\,a^{\varepsilon'\xi'}(\mathbf{p}'), \tag{5.9}$$

$$R^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\ \mathbf{p}')=\varphi(p,\ p')\{N_1 S^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\ \mathbf{p}') + N_2 T^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\ \mathbf{p}')\}, \tag{5.10}$$

where

$$\lambda = \frac{g^2}{8\pi^4},\ \varphi(p,\ p') = \frac{1}{2}\sqrt{\frac{(E+M)(E'+M)}{EE'\omega\omega'}}, \tag{5.11}$$

$$S^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\ \mathbf{p}') = \varepsilon\varepsilon'\xi\left(\frac{p}{E+M}\right)^{(1-\varepsilon)/2}\left(\frac{p'}{E'+M}\right)^{(1-\varepsilon')/2}[(W-M)\,m_{\varepsilon\varepsilon'}$$
$$+ n_{\xi\xi'}(\varepsilon E + \varepsilon' E' + \xi\omega + \xi'\omega' - M - W)] + \xi\left(\frac{E+M}{p}\right)^{(1-\varepsilon)/2}\left(\frac{E'+M}{p'}\right)^{(1-\varepsilon')/2}$$
$$\times \frac{\vec{\sigma}\mathbf{p}}{E+M}\,\frac{\vec{\sigma}\mathbf{p}'}{E'+M}[(W+M)\,m_{\varepsilon\varepsilon'} + (\varepsilon E + \varepsilon' E' + \xi\omega + \xi'\omega' + M - W)\,n_{\xi\xi'}],$$
$$T^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\ \mathbf{p}') = (\varepsilon + \xi)\left\{\frac{1}{W+M}\left(\frac{M-E}{p}\right)^{(1-\varepsilon)/2}\left(\frac{M-E'}{p'}\right)^{(1-\varepsilon')/2} +\right.$$
$$\left. + \frac{1}{W-M}\left(\frac{E+M}{p}\right)^{(1-\varepsilon)/2}\left(\frac{E'+M}{p'}\right)^{(1-\varepsilon')/2}\frac{\vec{\sigma}\mathbf{p}}{E+M}\,\frac{\vec{\sigma}\mathbf{p}'}{E'+M}\right\},$$
$$m_{\varepsilon\varepsilon'} = \{E_q(\varepsilon E_q + \varepsilon E + \varepsilon' E' - W)\}^{-1},$$
$$n_{\xi\xi'} = \{E_q(\xi E_q + \xi\omega + \xi'\omega' - W)\}^{-1}, \tag{5.12}$$

and E, E', ω, and ω' denote the energies of the nucleon and the meson with momentum $\mathbf{p}$ and momentum $\mathbf{p}'$, and $E_q = \sqrt{(\mathbf{p}+\mathbf{p}')^2 + M^2}$.

In accordance with what was said in Sec. 4, the asymptotic behavior of the function $a^{+,+}(\mathbf{p})$ in the scattering problems under consideration should correspond to the incident and outgoing waves. We accordingly put

$$a^{+,+}(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}_0) \tag{5.13}$$
$$+ f^{+,+}(\mathbf{p})\,\delta_+(E + \omega - W),$$

where, as usual, $\delta_+(x) = i\pi\delta(x) - (1/x)$ and for the remaining functions

$$\overset{+}{a}{}^{\varepsilon\xi}(\mathbf{p}) = \frac{1}{W - \varepsilon E - \xi\omega}\,f^{\varepsilon\xi}(\mathbf{p}). \tag{5.14}$$

Inserting Eqs. (5.13) and (5.14) into (5.9), we obtain

$$f^{\varepsilon\xi}(\mathbf{p}) = \frac{\lambda}{4\pi}\,R^{\varepsilon\xi}_{++}(\mathbf{p},\ \mathbf{p}_0) \tag{5.15}$$
$$+ i\frac{\lambda}{4}\int d\mathbf{p}'R^{\varepsilon\xi}_{++}(\mathbf{p},\ \mathbf{p}')\,f^{++}(\mathbf{p}')\,\delta(E' + \omega' - W)$$
$$+ \frac{\lambda}{4\pi}\sum_{\varepsilon'\xi'}\int d\mathbf{p}'\,\frac{R^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\ \mathbf{p}')\,f^{\varepsilon'\xi'}(\mathbf{p}')}{W - \varepsilon' E' - \xi'\omega'}.$$

To separate the angular variables, we employ the system of invariant functions $L_l^{\pm}$ for the spin and angles, as discussed in reference 14:

$$L_l^{+}(\mathbf{n},\ \mathbf{n}') = (l+1)\,P_l(\cos\theta) \tag{5.16}$$
$$- i\vec{\sigma}[\mathbf{n}\mathbf{n}']\,P_l^1(\cos\theta)\quad \text{for}\quad j = l + {}^1/_2,$$
$$L_l^{-}(\mathbf{n},\ \mathbf{n}') = lP_l(\cos\theta)$$
$$+ i\vec{\sigma}[\mathbf{n}\mathbf{n}']\,P_l^1(\cos\theta)\quad \text{for}\quad j = l - {}^1/_2.$$

Thanks to the transformation (5.8) introduced above, all that is needed to separate the angle variables is to expand the functions $f^{\epsilon\xi}$ and the kernels $R^{\epsilon\xi}_{\epsilon'\xi'}$ in polynomials of $L_l^{\pm}$. Introducing into Eq. (5.15) the expansion of the functions

$$f^{\varepsilon\xi}(\mathbf{p}) = \sum L_l^{\pm}\left(\frac{\mathbf{p}}{p},\frac{\mathbf{p}_0}{p_0}\right)f^{\varepsilon\xi}_{jl}(\mathbf{p}), \tag{5.17}$$

we obtain, after eliminating the angle and spin variables, a system of equations for the scattering amplitudes $f^{\epsilon\xi}_{jl}$, corresponding to the prescribed values of the total and orbital momentum, and depending only on the modulus of p:

$$f^{\varepsilon\xi}_{jl}(p) \tag{5.18}$$
$$= \frac{\lambda}{4\pi}\,{}^{jl}R^{\varepsilon\xi}_{++}(p,\ p_0)\left\{1 + i\lambda\,\frac{4\pi^2 p_0 E_0\omega_0}{E_0 + \omega_0}\,f^{++}_{jl}(p_0)\right\}$$
$$+ \lambda\sum_{\varepsilon'\xi'}\int\frac{p'^2dp'\ {}^{jl}R^{\varepsilon\xi}_{\varepsilon'\xi'}(p,\ p')}{W - \varepsilon' E' - \xi'\omega'}\,f^{\varepsilon'\xi'}_{jl}(p').$$

Here the kernels ${}^{jl}R$ are related to the functions ${}^{jl}S$ and ${}^{jl}T$ by the earlier Eq. (5.10), whereby, unlike Eq. (5.11), we have

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[14] I. E. Tamm, Iu. A. Gol'fand and V. Ia. Fainberg, J. Exper. Theoret. Phys. USSR 26, 649 (1954)

$$
\begin{aligned}
{}^{jl}S^{\varepsilon\xi}_{\varepsilon'\xi'} = \varepsilon\varepsilon'\xi \left(\frac{p}{E+M}\right)^{(1-\varepsilon)/2} \left(\frac{p'}{E'+M}\right)^{(1-\varepsilon')/2} & [\varepsilon(W-M)J_{k_1}(E+\varepsilon\varepsilon'E'-\varepsilon W) \\
& + \xi(\varepsilon E+\varepsilon'E'+\xi\omega+\xi'\omega'-M-W)J_{k_1}(\omega+\xi\xi'\omega'-\xi W)] \\
+ \xi\left(\frac{p}{E+M}\right)^{(1+\varepsilon)/2}\left(\frac{p'}{E'+M}\right)^{(1+\varepsilon')/2} & [\varepsilon(W+M)J_{k_2}(E+\varepsilon\varepsilon'E'-\varepsilon W) \\
& + \xi(\varepsilon E+\varepsilon'E'+\xi\omega+\xi'\omega'+M-W)J_{k_2}(\omega+\xi\xi'\omega'-\xi W)].
\end{aligned} \tag{5.19}
$$

We do not need the expression for ${}^{jl}T$, and we will not write it down. Equation (5.19) employs the notation

$$J_k(z) = \frac{1}{2}\int_{-1}^{+1}\frac{P_k(x)\,dx}{E_q(E_q+z)}, \tag{5.20}$$

$$E_q = \sqrt{M^2+p^2+p'^2+2pp'x},$$

where $P_k(x)$ are the Legendre polynomials. For the state $S_{1/2}$ ($j = 1/2$, $l = 0$) it is necessary to put in (5.18) $k_1 = 0$ and $k_2 = 1$; for the $P_{1/2}$ state we have $k_1 = 1$ and $k_2 = 0$, and for the $P_{3/2}$ state we have $k_1 = 1$ and $k_2 = 2$.

Equations (5.18) for the functions f_{jl} contain imaginary coefficients. However, with the aid of the transformation

$$U^{\varepsilon\xi}_{jl}(p) = f^{\varepsilon\xi}_{jl}(p)\left[1 + i\lambda\frac{4\pi^2 p_0\omega_0 E_0}{E_0+\omega_0} f^{++}_{jl}(p_0)\right]^{-1} \tag{5.21}$$

it is possible to obtain an equation with real coefficients for the real functions $U^{\epsilon\xi}_{jl}$. These equations differ from Eq. (5.18) only by the substitution of U for f and by the absence of an imaginary component in the brackets.

It is easy to verify that the phase of the scattering of the pi-mesons by nucleons for the waves j, l and I is given by the equation

$$\mathrm{tg}\,{}^{I}\delta_{jl} = -\frac{4\pi^2 p_0 E_0\omega_0}{E_0+\omega_0}\,{}^{I}U^{++}_{jl}(p_0). \tag{5.22}$$

The numerical values of the coefficients N_1 and N_2 in Eq. (5.10), relating R with S and T, depend on the isotopic spin I of the state, as shown in Eq. (5.5).

Let us note that according to Eq. (5.5) the coefficient N_2 in the function T differs from zero only when $I = 1/2$, and moreover, the coefficients ${}^{jl}T$ themselves, obtained by expanding T in polynomials of $L^{\pm}_{l}$, differ from zero only for the $S_{1/2}$ and $P_{1/2}$ states. This is explained by the fact that the kernel T corresponds to a "chain with absorption" Y_2, i.e., in the intermediate state we have only one nucleon at rest in the center-of-inertia system, and consequently having spins $J = I = 1/2$.

Unlike the kernel S, the isotopic behavior of the kernel T at large momenta is such, that those of Eqs. (5.18) containing the kernel T (that is, the equations for ${}^{1/2}S_{1/2}$ and ${}^{1/2}P_{1/2}$) have no finite solutions. This is due to the fact that iteration of the diagrams Y_1 and Y_2, corresponding to the kernels S and T and therefore taken into account in our integral equations, leads to diagrams containing peaks and overlapped by singularities of the self-energy type. Thus the equations for the states ${}^{1/2}S_{1/2}$ and ${}^{1/2}P_{1/2}$ must be subjected, in addition to renormalization of the self-energy terms, to an additional renormalization that eliminates the above singularities[11] Since we have not yet performed this additional renormalization, we are restricitng ourselves, like the authors of reference 2, to consideration of those Eqs. (5.18) that do not contain the kernel T. Let us note that all the equations considered in this section agree fully with the results of reference 2, provided we neglect all amplitudes and kernels for which ϵ and ξ differ from + 1.

When the energies W are not too large, only S and P waves play a considerable part in the scattering. At the present time we are engaged in solving numerically the system of Eq. (5.18) for the four states ${}^{3/2}S_{1/2}$, ${}^{3/2}P_{1/2}$, ${}^{3/2}P_{3/2}$ and ${}^{1/2}P_{3/2}$ at various energies (up to the energy corresponding to the kinetic energy of mesons in a laboratory system of the order of 300 mev). The results of the calculation will be published separately.

6. RESULTS OF RENORMALIZATION

Because of lack of space we cannot consider in detail the question of renormalization, and we shall therefore restrict ourselves to a brief statement of the results.

Inclusion of terms Y_1 and Y_2 in Eq. (5.1) leads to the appearance of additional "finite" (after renormalization) terms in the left halves of Eqs. (5.9), (5.15) and (5.18). As a result, for example, it is necessary to replace $f^{\epsilon\xi}$ in the left half of Eq. (5.18) by $\sum_{\varepsilon'\xi'}\Delta^{\varepsilon\xi}_{\varepsilon'\xi'} f^{\varepsilon'\xi'}$,

$$\Delta^{\varepsilon\xi}_{\varepsilon'\xi'} = b^{\varepsilon\xi}_{\varepsilon'\xi'}(W - \varepsilon'E - \xi'\omega)^{-1}, \tag{6.1}$$

where

$$b^{\varepsilon\xi}_{\varepsilon'\xi'} = \delta_{\varepsilon\varepsilon'}\delta_{\xi\xi'}\Big\{(W-\varepsilon E-\xi\omega)\,[1+A(\xi)] - \varepsilon\frac{M}{E}B(\xi) + \xi C(\varepsilon)\Big\} + \delta_{\varepsilon\varepsilon'}\delta_{\xi,-\xi'}\,\xi C(\varepsilon) + \delta_{\xi\xi'}\delta_{\varepsilon,-\varepsilon'}B(\xi)\frac{p}{E}; \tag{6.2}$$

$$A(\xi) = \frac{3}{2}\lambda\int_0^1 dU(1-U)\left\{\ln\left|\frac{\mu^2(1-U)+M^2U^2}{\mu^2(1-U)+\nu^2(U-U^2)+UM^2}\right| + \frac{2M^2U^2}{\mu^2(1-U)+M^2U^2}\right\}; \tag{6.3}$$

$$B(\xi) = \frac{3}{2}\lambda\int_0^1 dU\,U\ln\left|\frac{\mu^2(1-U)+M^2U^2}{\mu^2(1-U)+\nu^2(U-U^2)+UM^2}\right|, \tag{6.4}$$

$$C(\varepsilon) = -4\lambda\int_0^1 du\left\{[3\rho^2(U-U^2)+M^2]\ln\left|\frac{M^2+\rho^2(U-U^2)}{M^2-\mu^2(U-U^2)}\right| - \frac{(\rho^2+\mu^2)(U-U^2)[M^2-3\mu^2(U-U^2)]}{M^2-\mu^2(U-U^2)}\right\}, \tag{6.5}$$

$$\nu^2 = p^2-(W-\xi\omega)^2, \qquad \rho^2 = p^2-(W-\varepsilon E)^2. \tag{6.6}$$

For large values of p and for sufficiently small values of λ, the determinant of the matrix $\Delta^{\varepsilon\xi}_{\varepsilon'\xi'}$ has the following form

$$\Delta \approx \left(1-\frac{3}{4}\lambda\ln p\right)^2\left(1-\frac{11}{4}\lambda\ln p\right)^2, \tag{6.7}$$

that is, it can vanish. Δ can also vanish for large values of λ. Here, as λ increases, the value p for which the determinant vanishes decreases. The vanishing of Δ is closely related to the results of the investigations described in reference 15, where it is shown that both in electrodynamics and in meson dynamics the solution of the approximate equations for the Green's function leads to the appearance in the corresponding functions of additional poles that have no direct physical meaning (see also reference 10). All this points out the limited applicability of the approximate equations.

In general, however, taking the higher approximations into account can lead to a radical change in the asymptotic behavior of Green's function, or in our case, to a corresponding substantial change in the behavior of Δ. Let us remark that if we limit ourselves to consideration of the functions U^{++} and U^{-+}, and if we also ignore the polarization of the meson vacuum, we obtain

$$\Delta = \left\{1 + A(+) - \frac{MB(+)}{(W-\omega)^2-E^2}\right\}^2 + B^2(+)\frac{p^2-(W-\omega)^2}{[(W-\omega)^2-E^2]^2}. \tag{6.8}$$

This expression does not vanish if λ is not too large.

Translated by J. G. Adashko
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[15] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 95, 497, 773, 1177 (1954); 96, 261 (1954); A. A. Abrikosov, A. D. Galanin and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 97, 793 (1954)

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Multiple Production of Particles in the Collisions of High Energy Nucleons with Nuclei

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The collisions of high energy nucleons with nuclei are examined in a statistical theory of the multiple production of particles. The relation is calculated between the entropy of the nucleon-nucleus system (as determined by the initial stage of the collision), and the number of particles resulting when the system flies apart into individual particles. The entropy of the system is calculated using relativistic hydrodynamics. The dependence of the number of produced particles on the energy of the incident nucleon and on the atomic weight of the nucleus is found.

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1. IN the work of Fermi[1] and subsequently Landau[2] a statistical theory was developed for the multiple production of particles in collisions of nucleons of very high energy. Only the collisions of nucleons with nucleons were examined, while the collisions of nucleons with nuclei were not investigated. In experiments, it is precisely collisions with nuclei that are observed, and without a theoretical study (of this subject) it is, strictly speaking, impossible to compare theory and experiment. The aim of this paper is to clarify some problems connected with the interaction of nucleons with nuclei at very high energies.

Let us note at first that Landau's[2] theoretical treatment of nucleon-nucleon collisions can be divided into two parts: In the first part the dependence of the total number of produced particles on the energy of the initial nucleons is calculated. In the second part, the angular and energy distribution of particles is obtained. The calculation of the total number of particles, using general thermodynamic relations, is relatively easy. The second part of the problem requires the complex methods of relativistic hydrodynamics. The results obtained for the total number of particles are much more accurate than for the angular and energy distributions. Because of approximations in the solutions, only the order of magnitude of the latter can be obtained.

In this paper we will limit ourselves to the first part of the nucleon-nucleus problem --- the calculation of the total number of produced particles as a function of the incident nucleon energy and the atomic number of the struck nucleus.

Consider the derivation of the energy dependence of the total number of particles in the nucleon-nucleon problem[2]. The following considerations are used. At the moment of nucleon collision, a system arises in which the mean free path is small compared to the dimensions. Thus the expansion is hydrodynamic in nature; the number of particles in the system remains undetermined during the expansion process, and becomes fixed only at the moment the system flies apart into separate particles.

Since during the whole expansion the motion is adiabatic, the system entropy remains constant until the system flies apart into separate particles. At the moment the system falls apart, the entropy of each small region is proportional to the number of particles in it. Summing over all such regions we have for the entire system:

$$N = \text{const}\cdot S, \tag{1}$$

where N is the total number of particles, and S is the total system entropy. Since the entropy remains constant until the moment of disruption, it is sufficient to calculate it at the beginning of the collision, immediately after the condensation of the system. It follows that in the center of mass frame, all matter is at rest immediately after the collision. Let E' be the energy of the nucleons in the center of mass frame. The total system entropy is proportional to $\epsilon^{3/4}\, V$, where ϵ is the energy density and V the volume in which the energy is distributed (this, if the equation of state for the system is $p = \epsilon/3$). Since V, due to the Lorentz contraction, transforms back proportional to E', and since the energy E in the laboratory system is proportional to E'^2, we finally obtain:

$$S \sim E^{1/4}. \tag{2}$$

Taking account of Eq. (1), we have

$$N \sim E^{1/4}. \tag{3}$$

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[1] E. Fermi, Progr. Theor. Phys. 5, 570 (1950)

[2] L. D. Landau, Izv. Akad. Nauk SSSR 17, 51 (1953)

Let us now consider the collision of a nucleon with a nucleus. Since the distance between nucleons in the nucleus is of the order of the nuclear force range, it is necessary to examine the process of particle production in the entire nuclear volume traversed by the incident nucleon. It is not difficult to see that both assumptions used to derive Eq. (3) need modification in the nucleon-nucleus case. In the first place, due to the presence of the nucleons participating in the process it is impossible to consider the number of particles proportional to the entropy S. Secondly, the calculation of the entropy change must be modified, since the asymmetry of the collision makes it impossible to indicate a time after the collision when all matter is at rest.

2. We begin by finding the dependence of the number of particles produced on the entropy. Let the disruption of the system into single particles take place at a temperature T_k, of the order $m_\pi c^2$, where m_π is the mass of a π-meson, and c the velocity of light. Particles at this temperature can be considered to form a perfect gas. Since the temperature at disruption is unknown, it may be a relativistic gas. We shall present expressions for the number of particles and for the entropy of a Bose gas (π-mesons) and a Fermi gas (nucleons). We shall not consider other particles, since their contribution is relatively small.

For the density of nucleons, we have

$$n_n = \left(\frac{kT}{\hbar c}\right)^3 \frac{g_n}{2\pi^2} \left[F_1(z_1, y_1) + F_1(z_1, y_2)\right], \tag{4}$$

where

$$z_1 = \frac{Mc^2}{kT}, \qquad y_1 = \frac{\mu_1}{kT}, \qquad y_2 = \frac{\mu_2}{kT}.$$

Here g_H is the statistical weight, in this case four (two spin states and two charge states), M the nucleon mass, μ_1 and μ_2 chemical potentials, respectively, of nucleons and antinucleons [the second term of (4) is for anti-nucleons].

The density of π-mesons is

$$n_\pi = \left(\frac{kT}{\hbar c}\right)^3 \frac{g_\pi}{2\pi^2} F_2(z,0). \tag{5}$$

Here $\quad g_\pi = 3, \quad z = m_\pi c^2 / kT,$

$$F_{1,2}(z,y) = z^3 \int_0^\infty \frac{\exp\{-z\sqrt{1+x^2}+v\}}{1 \pm \exp\{-z\sqrt{1+x^2}+y\}}\, x^2 dx. \tag{6}$$

The plus sign (corresponding to the function F_1) is for fermions, the minus sign (corresponding to the function F_2) for bosons[3].

If $z > y$, then $F_{1,2}(z, y)$ may be put in series form:

$$F_{1,2}(z, y) = z^2 \sum_{m=0}^{\infty} (\mp 1)^m \frac{\exp\{y(m+1)\}\, K_2[z(m+1)]}{(m+1)}, \tag{7}$$

where $K_2(z)$ is a modified Bessel function of the second kind.

We now derive an expression for the particle entropy. For the density of entropy of nucleons and anti-nucleons, following from the relation $S = (E - N_\mu - \Omega)/T$, we have

$$S_n = k\left(\frac{kT}{\hbar c}\right)^3 \frac{g_n}{2\pi^2} \left[G_1(z_1, y_1) + G_1(z_1, y_2) - y_1 F_1(z_1, y_1) - y_2 F_1(z_1, y_2)\right]. \tag{8}$$

For the density of meson entropy we obtain

$$S_\pi = k\left(\frac{kT}{\hbar c}\right)^3 \frac{g_\pi}{2\pi^2} G_2(z, 0), \tag{9}$$

where

$$\begin{aligned} G_{1,2}(z, y) &= \Phi_{1,2}(z, y) - \Psi_{1,2}(z, y) \\ \Phi_{1,2}'(z, y) &= z^4 \int_0^\infty \frac{x^2\sqrt{1+x^2}\,dx}{\exp\{-y + z\sqrt{1+x^2}\} \pm 1} \\ \Psi_{1,2}(z, y) &= \mp z^3 \int_0^\infty \ln\left(1 \pm \exp\{y - z\sqrt{1+x^2}\}\right) x^2\, dx. \end{aligned} \tag{10}$$

The functions $\Phi_{1,2}(z, y)$[3] and $\Psi_{1,2}(z, y)$ are related to the energy and thermodynamic potential Ω of the particles by the following:

$$\varepsilon = \frac{E}{V} = kT \frac{g}{2\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \Phi_{1,2}(z, y); \tag{11}$$

$$\omega = \frac{\Omega}{V} = kT \frac{g}{2\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \Psi_{1,2}(z, y),$$

where ϵ and ω are the energy density and thermodynamic potential density respectively.

For the case $z < y$, $G_{1,2}(z, y)$ may be written as the following series:

$$G_{1,2}(z, y) = z^2 \sum_{m=0}^{\infty} e^{y(m+1)} (\mp 1)^m \times \frac{4K_2[z(1+m)] + z(1+m) K_1[z(1+m)]}{(1+m)^2}, \tag{12}$$

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[3] S. Z. Belen'kii, Dokl. Akad. Nauk SSSR 99, 523 (1955)

$z=\frac{m_\pi c^2}{kT}$ (m_π=mass of the π-meson)	$F_1(z,0)$	$F_2(z,0)$	$\Phi_1(z,0)$	$\Phi_2(z,0)$	$G_1(z,0)$	$G_2(z,0)$	$\alpha(z)$
0	1.803	2.40	5,68	6.49	7,57	8,65	0,25
0.5	1.72	2.17	5,58	6.30	7,37	8,31	0.213
0.7	1.65	2.02	5.47	6.12	7,19	8.02	0.216
0.9	1.56	1.86	5.33	5.90	6,95	7,67	0.221
1	1.52	1.78	5.24	5,78	6,81	7,48	0.223
1.2	1.41	1.62	5.05	5.51	6,51	7,07	0.222
1.5	1.25	1.39	4,72	5.06	6,00	6,42	0.215
2	0.982	1.05	4,07	4.27	5,07	5.31	0.198
3	0.546	0.561	2.72	2.78	3,27	3,33	0.169
6	0.0559	0.0599	0.471	0.471	0.531	0,531	0.113
7	0.0268	0.0268	0,237	0.237	0,263	0.263	0.102
8	0.0117	0.0117	0.115	0.115	0.127	0.127	0,0927

where $K_1(z)$ is a modified Bessel function of the first kind.

The functions $F_{1,2}(z,0)$, $\Phi_{1,2}(z,0)$, and $G_{1,2}(z,0)$ which determine the density of particles, energy and entropy for fermions and bosons are given in the Table.

Since nucleons and anti-nucleons annihilate in pairs, creating various particles with total chemical potential of zero, it follows that the chemical potential of an anti-nucleon is equal and opposite to that of a nucleon. That is, $y_1 = y$ and $y_2 = -y$.

From Eqs. (4)-(9) we obtain the following expression for the total entropy of the system:

$$\frac{S}{k} = \frac{G_1(z_1, y) + G_1(z_1, -y) - y\,[F_1(z_1, y) - F_1(z_1, -y)}{F_1(z_1, y) + F_1(z_1, -y)} N_{\text{н}} + \frac{G_2(z,0)}{F_2(z,0)} N_\pi. \tag{13}$$

Here N_n is the total number of nucleons and anti-nucleons, and N_π the total number of π-mesons in the system. We have made the transition from particle density to the total number with the assumption that each region of the system at disruption has the same average values of z and y. Furthermore, we take into account the conservation of nuclear charge, which implies that the difference between the number of nucleons N_{nn} and anti-nucleons N_{an} must be equal to the number of initial nucleons N_0

$$N_{nn} - N_{an} = \left(\frac{kT}{\hbar c}\right)^3 \frac{g_{\text{н}}}{2\pi^2}[F_1(z_1, y) - F_1(z_1, -y)]\, V = N_0. \tag{14}$$

Here V is the total volume of the system.

Let us first examine the case $N_0 = 0$. It follows from Eq. (14) that $y = -y = 0$. From Eq. (13) we obtain

$$N^* = \alpha \frac{S}{k},$$

where
$$\alpha(z) = \frac{2g_n F_1(z_1, 0) + g_\pi F_2(z, 0)}{2g_n G_1(z_1, 0) + g_\pi G_2(z, 0)} \tag{15}$$

and $N^* = N_m + N_\pi$, i.e., N^* is equal to the sum of particles produced with $N_0 = 0$. The function $\alpha(z)$ is given in Table I. It is evident from the Table that $\alpha(z)$ is weakly dependent on z, i.e., on the temperature.

It is not difficult to show that when $kT \ll m_\pi c^2$, $\alpha(z) \approx 1/(z + 5/2)$, i.e., $N^* \approx [1/(z + 5/2)] \times (S/k)$.

Let us now return to Eq. (13). Instead of S we use the ratio of the number of particles N^* (produced with $N = 0$) to N_0. Multiplying both sides of Eq. (13) by $\alpha(z)$ and dividing by N_0, we obtain

$$\frac{N^*}{N_0} = \left[\frac{G_1(z_1, y) + G_1(z_1, -y)}{F_1(z_1, y) - F_1(z_1, -y)} - y + \frac{g_\pi}{g_n}\frac{G_2(z, 0)}{F_1(z_1, y) - F_1(z_1, -y)}\right]\alpha(z). \tag{16}$$

On the other hand, it is not difficult to see that the total number of particles is

$$\frac{N}{N_0} = \frac{N_\pi + N_n}{N_0} = \frac{g_\pi F_2(z, 0) + g_n[F_1(z_1, y) + F_1(z_1, -y)]}{g_n[F_1(z_1, y) - F_1(z_1, -y)]}. \tag{17}$$

The two equations (16) and (17), together with the parameter y, determine the dependence of N/N_0 on N^*/N_0 (i.e., on the ratio of entropy to the initial number of nucleons). If we assume $z_1 \gg y$, and therefore leave only the first terms of the $F_1(z, y)$ and $G_1(z, y)$ series expansions, the solution of the equations becomes materially easier. Physically this assumption means that we neglect the difference between a Fermi and Maxwell distribution for the nucleons, which is allowable in this case. Since the quantity y then appears as an exponential in the distribution function, it is not difficult to show that

$$F_1(z_1, y) - F_1(z_1, -y) = 2F_1(z_1, 0)\,\mathrm{sh}\, y,$$

$$F_1(z_1, y) + F_1(z_1, -y) = 2F_1(z_1, 0)\,\mathrm{ch}\, y,$$

$$G_1(z_1, y) + G_1(z_1, -y) = 2G_1(z_1, 0)\,\mathrm{ch}\, y.$$

As a result, instead of Eqs. (16) and (17), we have

$$\frac{N^*}{N_0} = \left[\frac{G_1(z_1, 0)}{F_1(z_1, 0)}\,\mathrm{cth}\, y - y + \frac{g_\pi G_2(z, 0)}{2g_n F_1(z_1, 0)\,\mathrm{sh} y}\right]\alpha(z), \tag{18}$$

$$\frac{N}{N_0} = \frac{N_\pi}{N_0} + \mathrm{cth}\, y,$$

$$y = \mathrm{Arsh}\left[\frac{g_\pi F_2(z, 0)}{2g_n F_1(z_1, 0)}\frac{N_0}{N_\pi}\right].$$

From these equations the ratio N/N_0 as a function of N^*/N_0 was found. This is shown in Fig. 1.

If the number of initial nucleons is sufficiently large, and if the critical temperature T_k is not too high, then $\sinh y = \dfrac{g_\pi F_2(z)}{2g_n F_1(z)}\dfrac{N_0}{N_\pi}$ is much larger than unity. This condition means that the number of initial nucleons significantly surpasses the number of nucleons produced with $N_0 = 0$.

If sinh y and $y > 1$, then Eq. (18) becomes

$$\gamma = \delta + \alpha(z)\ln(\delta - 1) + B(z), \tag{19}$$

where
$$\gamma = \frac{N^*}{N_0}, \qquad \delta = \frac{N}{N_0},$$

$$B(z) = \alpha(z)\left[\frac{G_1(z_1, 0)}{F_1(z_1, 0)} - \ln\left(\frac{g_\pi F_2(z, 0)}{g_n F_1(z_1, 0)}\right)\right] - 1.$$

With $z \gg 1$

$$B(z) = \frac{\ln(g_n/g_\pi)(M/m)^{3/2}}{z + 5/2}, \quad z = \frac{m_\pi c^2}{kT}.$$

If $\gamma > 1$, then $\gamma \approx \delta$; with $\delta - 1 < 1$

$$\delta - 1 = \exp\left\{-\frac{B(z) + 1 - \gamma}{\alpha(z)}\right\};$$

if also $z \gg 1$, then

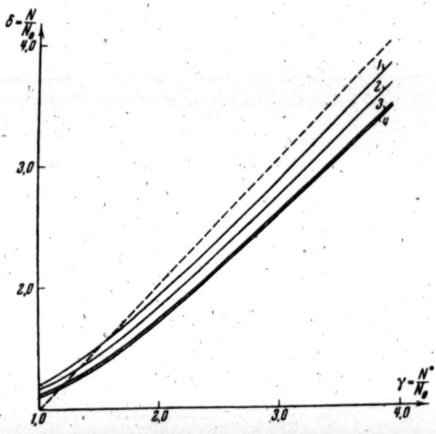

FIG. 1. kT/mc^2 is: *1.* 1.25; 2. 1.0; 3. 0.67; 4. 0.5. The dotted line corresponds to $\gamma = \delta$.

$$\frac{N-N_0}{N_0}=\delta-1\simeq\frac{g_n}{g_\pi}\left(\frac{M}{m}\right)^{3/2}$$

$$\times\ \exp\left\{-\left(\frac{mc^2}{kT}+\frac{5}{2}\right)\left(1-\frac{N^*}{N_0}\right)\right\}.$$

Figure 1 [calculated from Eq. (18)] shows, for the various chosen temperatures of disruption T_k, that N/N_0 is always less than N^*/N_0 until the $\gamma \approx 2$ region. (Note that N includes the number of initial nucleons N_0.) The difference between N/N_0 and N^*/N_0 is larger, the lower the temperature T_k. Already at $N^*/N_0 = 3$ and $z = mc^2/kT = 2$, $N/N_0 = 2.54$, i.e., the ratio $N/N^* = 0.85$. For $\gamma < 2$, the number of newly produced particles falls rapidly, and approaches unity. It is seen from this that the relation

$$N = \mathrm{const}\cdot S$$

is also valid for the collision of a nucleon with a nucleus until N^*/N_0 is of the order of two. N is the sum of particles produced during the collision and the initial nucleons.

3. Let us now calculate the change in entropy. In the nucleon nucleon problem it was not necessary to examine the mechanism of compression, since the results could be obtained immediately from symmetry considerations. The situation is quite different for the nucleon-nucleus collision. Let us apply our model of nuclear matter as a continuous medium to the first stage of the collision — the compression.

It will turn out from the following that the most convenient system of coordinates is that in which the nucleon and nucleus have equal and oppositely directed velocities. In this reference frame, due to the Lorentz contraction, both nucleon and nucleus look loke very thin disks. Thus, the problem can be considered one dimensional. In this case the collision of the nucleon with the nucleus appears as the collision of the nucleon with a tube cut from the nucleus[4], with a cross section equal to that of the nucleon and a length between the contracted diameters of nucleus and nucleon. With the close approach of the nucleon to the tube, impact waves propagate in both directions with velocity D through the nuclear and nucleon matter (Fig. 2). Of course, we can speak of the propagation of impact waves in the nucleon only provisionally. Hydrodynamic considerations are used in this case for orientation.

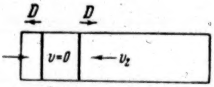

FIG. 2

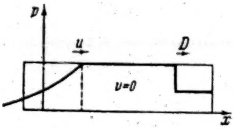

FIG. 3. The dotted line is the boundary of the rarefaction wave.

Because of the equality of velocities v_2 of the approaching particles in the chosen reference frame, the matter between the impact waves is at rest. The impact wave traveling to the left will reach the edge sooner than the wave traveling to the right (in contrast to the nucleon-nucleon situation where both waves reach an edge at the same time). When the impact wave reaches the left edge, the compression ceases, and an outflow of matter begins (Fig. 3). This means that a wave of refraction travels to the right with a velocity equal to that of sound in the medium. At the same time, the impact wave continues traveling to the right, since it has not yet reached the right edge. The calculation of the entropy will be different depending on whether the rarefaction wave overtakes the impact wave before it reaches the right edge. It is known that the speed of sound in a medium (equation of state $p = \epsilon/3$) is[5]

$$u = c/\sqrt{3}. \tag{20}$$

Let us now calculate the velocity of the impact wave. We change to a coordinate system in which the impact wave is stationary. Then (Fig. 2) the velocity of matter behind the impact wave is D, while ahead of the wave it is

$$v_2' = \frac{v_2 + D}{1+(v_2D/c^2)}.$$

Because of the continuity of energy and momentum flows through the impact wave front, we have[5,6]

$$\frac{p_1+(D^2/c^2)\,\varepsilon_1}{1-(D^2/c^2)} = \frac{p_2+(v_2'^2/c^2)\,\varepsilon_2}{1-(v_2'/c^2)}, \tag{21}$$

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[4] I.L. Rozental' and D.S. Chernavskii, Usp. Fiz. Nauk 52, 185 (1954)

[5] L. D. Landau and E. M. Lifshitz, *Mechanics of Continuous Media*, 1954

$$\frac{(D/c)(p_1+\varepsilon_1)}{1-(D^2/c^2)} = \frac{(v_2'/c)(p_2+\varepsilon_2)}{1-(v_2'^2/c^2)}. \qquad (22)$$

Here p_1 and ϵ_1 are the pressure and energy density behind the impact wave, and p_2 and ϵ_2 the pressure and energy density ahead of the wave. Dividing the first equation by the second, we obtain

$$\frac{p_1+(D^2/c^2)\varepsilon_1}{(D/c)(p_1+\varepsilon_1)} = \frac{p_2+(v_2'^2/c^2)\varepsilon_2}{(v_2'/c)(p_2+\varepsilon_2)}. \qquad (23)$$

Since the velocity v_2 of the colliding particles is very close to that of light, the velocity $v_2^{1\prime}$ will be also. Assuming $v_2^{1\prime} = c$ we see that the right side of Eq. (23) is unity, and consequenlty our results are independent of the equation of state for the matter ahead of the impact wave.

Making use of the equation of state for the matter behind the impact wave ($p_2 = \epsilon_2/3$), we obtain the following for D:

$$\left(\frac{1}{3}+\frac{D^2}{c^2}\right)\Big/\frac{4}{3}\frac{D}{c} = 1,$$

from which it follows that

$$D = {}^1/_3\, c. \qquad (24)$$

It is now possible to find the minimum tube length l_k in which the rarefaction wave will overtake the impact wave. This is determined by the ratio

$$\frac{(i_k/d)-1}{D} = \frac{(l_k/d)+1}{u} \quad \text{or} \quad \frac{l_k}{d} = \frac{D+u}{u-D},$$

where d is the nucleon "diameter".

Substituting the values of u and D, we obtain

$$l_k/d = 3.7. \qquad (25)$$

Let us assume that the tube length l is smaller than l_k, i.e., that the impact wave traveling to the right reaches the edge before it is overtaken by the rarefaction wave. In this case it is very easy to calculate the entropy. One calculates the entropy of the separate regions of the system immediately after the passage of the impact wave, at which time they are at rest in our reference frame. It is not difficult to see that the change in entropy of the entire system is then

$$\frac{S}{S_0} = \frac{1}{2}\left(\frac{l}{d}+1\right) \quad \text{or} \quad \frac{l}{d} \leqslant 3.7, \qquad (26)$$

where S_0 is the change of entropy in the nucleon-nucleon process, and l the tube length. This result may be obtained with the aid of a calculation applicable to the nucleon-nucleon problem (see above), keeping in mind that the effective volume is that of the tube plus that of the nucleon, and that the matter is at rest in the frame where velocities of the initial particles are equal and opposite. In an outline of such a calculation[4] the assumption is made that matter is at rest in the center of mass frame, but this is incorrect.

For tube lengths exceeding $l_k = 3.7$ the solution becomes more complex. The rarefaction wave overtakes the impact wave in this case, but cannot cross its wave front, since the impact wave travels in the matter of the oncoming nucleus with a velocity greater than that of sound in the matter. The rarefaction wave is reflected by the impact wave. A region is formed bounded by the impact wave on the right, and by the rarefaction wave on the left (Fig. 4).

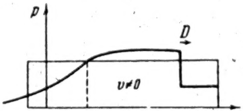

FIG. 4

To describe the motion of the medium in this region, we refer to the arbitrarily similar motion ~~of a~~ compressible gas[5]. In the ultra-relativistic case the equation of motion becomes[6]

$$3\frac{\partial^2\chi}{\partial\eta} - \frac{\partial^2\chi}{\partial y^2} - 2\frac{\partial\chi}{\partial y} = 0. \qquad (27)^*$$

Independent variables here are the quantities $\eta = \operatorname{arctanh} v/c$, where v is the velocity of the medium, c the velocity of light, and $y = \ln(T/T_0)$, T the temperature of the medium, T_0 the temperature at points where $v = 0$. The coordinate x and time t are expressed as a function of χ by

$$x = e^{-y}\left(\frac{\partial\chi}{\partial y}\operatorname{sh}\eta - \frac{\partial\chi}{\partial\eta}\operatorname{ch}\eta\right), \qquad (28)$$

$$t = e^{-y}\left(\frac{\partial\chi}{\partial y}\operatorname{ch}\eta - \frac{\partial\chi}{\partial\eta}\operatorname{sh}\eta\right).$$

Thus if the function $\chi(\eta, y)$ is found, Eq. (28) gives x and t as functions of T and v.

The region of interest is bounded on one side by the rarefaction wave, and on the other by the impact wave. Let us determine the boundary conditions for the function χ. It follows from reference 6 that, at the rarefaction wave boundary,

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* In reference 6, the coefficients are different because of typographical errors.

[6] I. M. Khalatnikov, J. Exper. Theoret. Phys. USSR 26, 529 (1954)

$$\chi = 0 \qquad \text{for} \qquad \eta = \sqrt{3}\, y. \tag{29}$$

must be satisfied.

Let us consider the boundary condition at the impact wave. Since in this case the matter behind the impact wave is not at rest, transformation to a reference frame in which the wave front is stationary gives for the velocity behind the impact wave:

$$v_1 = \frac{v_1 + D}{1 + (v_1 D / c^2)}. \tag{30}$$

Here v_1 is the velocity behind the impact wave in the reference frame where the incident particle velocities are equal.

Equations (21) - (24) remain applicable to the present case if v_1' is substituted for D. It follows that $v_1' = 1/3\, c$ [see Eq. (24)]. Equation (30) then gives a relation between v_1 and D which may be written

$$D = \frac{dx}{dt} = \frac{1 + 3\operatorname{th}\eta}{3 + \operatorname{th}\eta}, \tag{31}$$

since $v_1 = -\tanh\eta$. Using Eqs. (21) - (24) with v_1 substituted for D, it is not difficult to obtain the following expression which is obeyed on the impact wave:

$$\frac{\varepsilon}{\varepsilon_0} = \left(\frac{T}{T_0}\right)^4 = \frac{1 - (v_1/c)}{1 + (v_1/c)}.$$

In the variables η and y

$$\left(v_1 = -\operatorname{th}\eta, \quad y = \ln\frac{T}{T_0}\right);$$

this means

$$\eta = 2y. \tag{32}$$

Substituting the value of dx/dt calculated from Eq. (28) into Eq. (31), and using Eq. (32), we obtain the following condition on the impact wave:

$$\left(3\frac{\partial}{\partial y} + 5\frac{\partial}{\partial\eta}\right)\left(1 - \frac{\partial}{\partial y}\right)\chi = 0 \quad \text{for } \eta = 2y. \tag{33}$$

Let us change to the variables

$$\alpha = \eta - 2y, \quad \beta = \sqrt{3}\, y - \eta. \tag{34}$$

In the new variables the equations for χ and the boundary conditions take the following form:

$$\frac{\partial^2\chi}{\partial\alpha^2} - 2\sqrt{3}(2-\sqrt{3})\frac{\partial^2\chi}{\partial\alpha\,\partial\beta} - 4\frac{\partial\chi}{\partial\alpha} + 2\sqrt{3}\frac{\partial\chi}{\partial\beta} = 0; \tag{35}$$

$$\chi = 0 \text{ when } \beta = 0,$$

$$\left[\frac{\partial}{\partial\alpha} + (5 - 3\sqrt{3})\frac{\partial}{\partial\beta}\right] \times \left[1 + 2\frac{\partial}{\partial\alpha} - \sqrt{3}\frac{\partial}{\partial\beta}\right]\chi = 0 \quad \text{when} \quad \alpha = 0. \tag{36}$$

Let us apply a Laplace transformation to the variable β

$$f(\alpha, p) = \int_0^\infty \chi(\alpha, \beta)\, e^{-p\beta}\, d\beta \tag{37}$$

and search for a solution of $f(\alpha, p)$ in the form $a(p)e^{k\alpha}$. Then Eqs. (35) and (36) yield the following algebraic equations for $a(p)$ and k:

$$k^2 - 2\sqrt{3}(2-\sqrt{3})kp - 4k + 2\sqrt{3}\,p = 0, \tag{38}$$

$$[k + (5 - 3\sqrt{3})p][1 + 2k - \sqrt{3}\,p]\,a = (9 - 5\sqrt{3})\lambda$$

Here λ is the coordinate value at which the rarefaction wave overtakes the impact wave;

$$\lambda = \left(\frac{\partial\chi}{\partial\beta}\right)_{\substack{\alpha=0\\ \beta=0}} = -\left(\frac{\partial\chi}{\partial\eta}\right)_{\substack{\eta=0\\ y=0}}. \tag{39}$$

Having determined $a(p)$ and k, an inverse Laplace transform would give χ, and thus solve the problem. However, we are not interested in the complete solution, but rather in the change of entropy during the time that the impact wave passes. The entropy change can be written:

$$S_1 = \sigma_0 \int_{t_1'}^{t_2'} s u_1\, dt'. \tag{40},$$

Here σ_0 is the tube cross section, and s is the entropy density after the impact wave; $u_1 = \frac{(v_1'/c)}{\sqrt{1-(v_1'/c)^2}}$, where v_1' is the matter velocity behind the impact wave front (since $v_1'/c = 1/3$, $u_1 = 1/2\sqrt{2}$), t_1' is the time at which the rarefaction wave overtakes the impact wave, t_2' the time at which the impact wave reaches the right edge and dt' an element of time in the reference frame where the impact wave front is at rest:

$$dt' = dt\,\sqrt{1-D^2} = \sqrt{dt^2 - dx^2}.$$

Using Eq. (28), the condition $\eta = 2y$ at the impact wave, and the fact that the ratio of entropy densities s/s_0 behaves as $(T/T_0)^3$ when the matter after the impact wave comes to rest, we obtain

$$S_1 = \sigma_0 \frac{1}{9} \int_0^{y_k} e^{2y} \left[\frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} - 1 \right) \chi \right]_{\eta = 2y} dy. \qquad (41)$$

Here y_k is the value of y at the instant that the impact wave reaches the edge of the system.

Changing to the variables α and β we have

$$S_1 = \sigma_0 \frac{1}{9(2-\sqrt{3})} \int_0^{\beta_k} e^{-2\beta/(2-\sqrt{3})} \qquad (42)$$

$$\times \left[\left(\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta} \right) \left(1 + 2\frac{\partial}{\partial \alpha} - \sqrt{3}\frac{\partial}{\partial \beta} \right) \chi \right]_{\alpha=0} d\beta.$$

Multiply both sides by $\exp\left\{\frac{2\beta_k}{2-\sqrt{3}}\right\}$. Let us then carry out a Laplace transformation. Using the curl theorem, we obtain for a representation of the quantity $S_1 \exp\left\{\frac{2\beta_k}{2-\sqrt{3}}\right\}$:

$$\Psi(p) = \int_0^{\infty} S_1 \exp\left\{\frac{2\beta_k}{2-\sqrt{3}}\right\} e^{-p\beta}\, d\beta$$

$$= \frac{\sigma_0}{9(2-\sqrt{3})} \frac{1}{p - 2/(2-\sqrt{3})}$$

$$\times \{(k-p)(1+2k-\sqrt{3}p)a - \sqrt{3}\lambda\}.$$

Substitute into this the values of k and a calculated from Eq. (38). (For k a quadratic equation is obtained; the solution with the minus sign before the square root is chosen.) As a result, we have

$$\Psi(p) = \sigma_0 \left[\frac{(5+3\sqrt{3})\lambda}{2\sqrt{3}(q-2)(q-5)} \right. \qquad (43)$$

$$\left. + \frac{(\sqrt{3}-1)\lambda}{2\sqrt{3}(q-2)(q-5)} \frac{1}{\sqrt{3}(q-1) + \sqrt{3(q-1)^2+1}} \right],$$

where $q = (2-\sqrt{3})p$.

Inverting the Laplace transform, we have

$$S_1 \exp\{2\beta_k/(2-\sqrt{3})\} \qquad (44)$$

$$= \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \Psi(p)\, e^{p\beta}\, dp.$$

The integral in Eq. (44) is taken in the complex plane along a line which is to the right of all poles of the integrand.

Substituting Eqs. (43) into (44), the integration is carried out. We make use of the known relation[7]

$$\frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{dp\, e^{tp}}{p + \sqrt{p^2+1}} = \frac{J_1(t)}{t}.$$

As a result we obtain:

$$S_1 = \sigma_0 \left[\frac{5+3\sqrt{3}}{6\sqrt{3}} s_0 \lambda (e^{3\beta'_k} - 1) \right. \qquad (45)$$

$$\left. + \frac{\sqrt{3}-1}{6\sqrt{3}} s_0 \lambda e^{-3\beta'_k} \int_0^{\beta'_k} (e^{4z} - e^{z}) \frac{J_1((\beta'_k - z)/\sqrt{3})}{\beta'_k - z} dz \right],$$

where $\beta'_k = \beta_k/(2-\sqrt{3})$.

To determine β_k we have the following relation:

$$x_k + t_k - t_0 = L, \qquad (46)$$

where t_0 is the instant at which the rarefaction wave overtakes the impact wave, L the coordinate value of the right edge at this instant, x_k and t_k the coordinate and time when the impact wave reaches the edge of the tube. Expressing x and t through a potential, and remembering that $t_0 = \lambda\sqrt{3}$, we obtain

$$\frac{\partial \chi}{\partial y} - \frac{\partial \chi}{\partial \eta} = (L + \sqrt{3}\lambda)\, e^{-y_k} \qquad (47)$$

or in the variables α and β:

$$-3\left[\frac{\partial \chi}{\partial \alpha} - (\sqrt{3}-1) \frac{\partial \chi}{\partial \beta'_k} \right]_{\alpha=0} \qquad (47')$$

$$+ 8\left(\frac{\partial \chi}{\partial \beta'_k} \right)_{\alpha=0} = (L + \sqrt{3}\lambda)\, e^{\beta_k}.$$

A similar calculation, carried to higher order, leads to the following expression:

$$(7 + 4\sqrt{3})\, e^{4\beta_k} - 1 \qquad (48)$$

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[7] V. A. Ditkin and I. I. Kuznetsov, *Handbook of Operational Calculus*, Moscow, 1951

$$+\int_0^{\beta_k}(e^{4z}-1)\frac{J_1[(\beta_k'-z)/\sqrt{3}]}{\beta_k'-z}dz=\frac{6(L+\sqrt{3}\lambda)}{\sqrt{3}(\sqrt{3}-1)\lambda}.$$

The terms containing integrals in Eqs. (45) and (48) contribute not more than 2% ot the total, and may be thus dropped. The coefficient of the first term in Eq. (45) $(5+3\sqrt{3})/6\sqrt{3}$ can be replaced by one to a good degree of approximation. As a result we have

$$S_1=\sigma_0\lambda s_0(e^{3\beta_k'}-1), \tag{49}$$

$$(7+4\sqrt{3})e^{4\beta_k'}$$

$$=\frac{\sqrt{3}(\sqrt{3}+1)}{\lambda}(L+\sqrt{3}\lambda)+1.$$

It is not difficult to be convinced that $L+\sqrt{3}\lambda = 4l-2d$. Since $\lambda=(3+\sqrt{3})d$, the second of Eq. (49) may be written:

$$(7+4\sqrt{3})e^{4\beta_k'}=4\frac{l}{d}-1.$$

Finally, we have for the change in entropy the following expression :

$$\frac{S}{S_0}=0.92\left(\frac{l}{d}-\frac{1}{4}\right)^{3/4}\quad\text{with}\quad\frac{l}{d}\geqslant 3.7. \tag{50}$$

where S_0 is the change in entropy due to the collision of the nucleon with the nucleus, S is the whole change of entropy due to the collision of the nucleon with a tube of length l.

In head on collisions of a nucleon and nucleus $l/d = A^{1/3}$, where A is the atomic number. The value $l = 3.7$ corresponds to $A = 51$. If Eqs. (26) and (50) are averaged over all possible collisions in the nucleus, from head on, to collisions of the incident nucleon with peripheral nucleons, we obtain (not separately considering lateral nucleon-nucleon collisions, which, we feel, are already accounted for in the equation for nucleon-nucleon collisions):

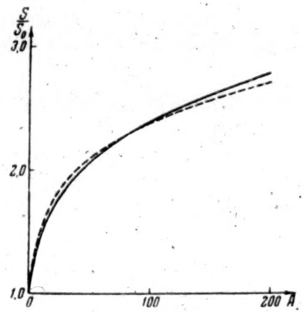

FIG. 5. The dotted line is $S/S_0 = A^{0.19}$.

For $A < 51$

$$\frac{S}{S_0}=\frac{1}{3}\frac{A-(2A^{1/3}-1)^{3/2}}{(A^{1/3}-1)^2}+0.5; \tag{51}$$

For $A > 51$

$$\frac{S}{S_0}=\frac{4}{(A^{1/3}-1)^2}\Big[0.167(A^{11/12}-A_0^{11/12}) \tag{52}$$

$$+\frac{1}{12}(A_0-(2A^{1/3}-1)^{3/2})\Big]$$

$$-0.6\frac{A^{2/3}-A_0^{2/3}}{(A^{1/3}-1)^2}+0.5,\ \text{where } A_0=51.$$

Figure 5 shows the dependence of S/S_0 on A. The dependence may be approximated with an accuracy of 4% by the expression

$$S/S_0=A^{0.19}. \tag{53}$$

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On the Theory of Nucleons

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A study is made of the hypothesis that the pion-nucleon interaction is indirect (transmitted by an intermediary field). This leads to a representation of the nucleon as an inner core surrounded by a pion cloud.

1. INTRODUCTION

THE experiments on pion-nucleon scattering[1-3] show clearly the existence of so-called isobars, i.e., excited states of nucleons. Also the peculiar decay of the Λ^0 – particle compels us to consider it as an excited state of the proton[4].

It is therefore very probable that at the present time we have already observed two levels of the proton, one with a short life-time which is responsible for the meson scattering, and one with a long life-time which is observed as the Λ^0 - particle.

In this connection there have recently appeared some papers[5,6] treating isobaric states of nucleons phenomenologically, without penetrating into the internal structure of the nucleon. A theory describing isobaric states as manifestations of the internal dynamics of the nucleon was worked out long ago, namely, the "strong-coupling" theory of Wentzel, Pauli and Dancoff[7]. The chief obstacle to the development of this theory was its lack of relativistic invariance, arising from the "smearing-out" of the nucleon by a hypothetical form-factor. Because of the presence of such a form-factor $K(\mathbf{x} - \mathbf{x}') \neq \delta(\mathbf{x} - \mathbf{x}')$, signals are transmitted within the nucleon with greater than light velocity.

One way and another, all the existing proposals for escaping from infinite particle self-energies run up against the same fundamental difficulty. For example, up to now nobody has found a logically consistent formulation of a non-local theory[8]. In the development of non-linear theories[9,10], there are also fundamental difficulties; the infinite zero-point energy appears non-additively in the energies of excited states of a non-linear field, and the attempt to quantize the field by means of a discrete lattice-space[11] leads again, as it is easy to show, to signals propagating with infinite velocity.

Our purpose in the present work is to explain a point of view which justifies the introduction of a form-factor in strong-coupling theory, while remaining within the framework of contemporary quantum field theories. The questions connected with infinite particle self-energies are not resolved but are circumvented. The essence of our proposal is that the nucleon interacts with the pi-meson not directly but only through a hypothetical K-meson with a large mass M.

There are two ways of looking at this model. One may say that a short-range force with a range (h/Mc) is acting between nucleon and pion. Or one may say that the nucleon has a composite structure, consisting of a small core of size (h/Mc) and a pion cloud, just as an atom consists of a nucleus and an electron cloud. Furthermore, just as a relativistic theory of the electronic shell of an atom can be constructed without solving the problems of nuclear structure, a relativistic theory of the pion shell around a nucleon ought to be possible without a complete theory of the core. In what follows we present an outline of such a theory.

2. CALCULATION OF THE INTERACTION BETWEEN NUCLEON AND PION FIELD

We postulate the existence of K-mesons with mass M and charge $\pm e$ and 0. Their wave-function is denoted by θ_s ($s = 1, 2, 3$ corresponding to the

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charges $\pm e$, 0). For definiteness we suppose θ_s to be a pseudoscalar.

The pion wave-function we represent in the usual way by ϕ_s (s = 1, 2, 3), the nucleon wave-function by ψ. The interactions between K- and π-meson, and between K-meson and nucleon, cannot at this stage be uniquely determined. Since we are only explaining the general structure of the theory, it is reasonable to start with the simplest assumptions about the form of these interactions.

We suppose the interactions to have the form

$$W = \int dx \left[g \sum_{s=1}^{3} \theta_s \varphi_s + \frac{\sqrt{4\pi} f}{\varkappa} \sum_{s=1}^{3} \psi^+ \gamma_\mu \gamma_5 \tau_s \psi \frac{\partial \theta_s}{\partial x_\mu} \right]. \tag{1}$$

Here g and f are two coupling constants, $\kappa = Mc/h$, τ_s are the isotopic spin matrices and γ_μ, γ_5 are the Dirac matrices. Thus we assume a charge-symmetric scalar interaction between K- and π-meson, and a charge-symmetric pseudovector interaction between K-meson and nucleon. The complete Hamiltonian of the system is

$$H = H_1^0 + H_2^0 + H_3^0 + W = H^0 + W, \tag{2}$$

where H_1^0, H_2^0 and H_3^0 are the Hamiltonians for free nucleons, pions and K-mesons, respectively. The theory with the Hamiltonian (2) is obviously relativistically invariant, but it is not free from self-energy divergences produced by the (θ, ψ) and (θ, ϕ) interactions. However, these divergences lie at a deeper level, since they are related to the nucleon core and not to the pion cloud.

To find the pion-nucleon interaction, we try to eliminate the K-meson field from (2). We do this by means of a chain of Tamm equations. Let Ω be the wave-functional, depending on the fields θ, ϕ and ψ. For the variables describing the field θ, we take the number N_q^s of K-mesons in the charge-state s with momentum q. We expand the field θ in a Fourier series

$$\theta_s = \sum_q \frac{1}{\sqrt{2\varepsilon_q}} \left[\theta_s(q) e^{i\mathbf{qx}} + \theta_s^+(q) e^{-i\mathbf{qx}} \right], \tag{3}$$

where $\theta_s(q)$ is an annihilation operator and $\theta_s^+(q)$ is a creation operator for a K-meson of energy ϵ_q. Further, we write Ω (0) for a state without K-mesons, $\Omega(1_q^s)$ for a state with one K-meson (s, q), etc. The equation

$$(H_0 - E)\,\Omega = -W\Omega \tag{4}$$

then expands into the following chain of equations:

$$(H_0 - E)\,\Omega(0) = -\left\{ \int d\mathbf{x} \left[g \sum_{s=1}^{3} \varphi_s(x) \sum_q \frac{1}{\sqrt{2\varepsilon_q}} e^{-i\mathbf{qx}} \right.\right. \tag{5}$$

$$\left.\left. - i \frac{\sqrt{4\pi} f}{\varkappa} \sum_{s=1}^{3} D_\mu^s(x) \sum_q \frac{q_\mu}{\sqrt{2\varepsilon_q}} e^{-i\mathbf{qx}} \right]\right\} \Omega(1_q^s),$$

$$(H_0 + \varepsilon_q - E)\,\Omega(1_q^s) = -\left\{ \int d\mathbf{x}' \left[g\varphi_s(x') \frac{2}{\sqrt{2\varepsilon_q}} e^{i\mathbf{qx}'} \right.\right.$$

$$\left.\left. + i \frac{\sqrt{4\pi} f}{\varkappa} D_\mu^s(x') \frac{q_\mu}{\sqrt{2\varepsilon_q}} e^{i\mathbf{qx}'} \right]\right\} \Omega(0) + \cdots,$$

where $D_\mu^s(x) = \psi^+ \gamma_\mu \gamma_5 \tau_s \psi$. Denoting the operator $(H_0 + \epsilon_q - E)^{-1}$ by Δ^{-1} (it is approximately equal to ϵ_q^{-1}) we obtain

$$(H_0 - E)\,\Omega(0)$$

$$= -\left\{ \iint d\mathbf{x}\, d\mathbf{x}' \left[g^2 \sum_{s=1}^{3} \varphi_s(x) \varphi_s(x') K(\mathbf{x} - \mathbf{x}') \right.\right. \tag{6}$$

$$+ \frac{4\pi f^2}{\varkappa^2} \sum_{s=1}^{3} D_\mu^s(x) D_\nu^s(x') \frac{\partial^2 K(\mathbf{x} - \mathbf{x}')}{\partial x_\mu \partial x'_\nu}$$

$$\left.\left. + \frac{2\sqrt{4\pi} fg}{\varkappa} \sum_{s=1}^{3} D_\mu^s(x) \varphi_s(x') \frac{\partial K(\mathbf{x} - \mathbf{x}')}{\partial x'_\mu} \right]\right\} \Omega(0),$$

with

$$K(\mathbf{x}-\mathbf{x}') = \int \left(e^{i\mathbf{q}(\mathbf{x}-\mathbf{x}')}/2\epsilon_q^2\right) d\mathbf{q} \qquad (7)$$

where we use the approximation $\Delta^{-1} \approx \epsilon_q^{-1}$. The differentiation of $K(\mathbf{x}-\mathbf{x}')$ with respect to x_4' is to be first performed with $x_4' \neq x_4$, and then x_4' is to be set equal to x_4. When $x_4' = x_4$ the function $K(\mathbf{x}-\mathbf{x}')$ becomes

$$K(\mathbf{x}-\mathbf{x}') = e^{-\varkappa r}/r, \qquad r = |\mathbf{x}-\mathbf{x}'|. \qquad (7')$$

Equation (6) is an approximate equation for $\Omega(0)$. The first term in the square bracket, proportional to g^2, gives an interaction of pions among themselves. The second term gives a short-range interaction between nucleon cores. The third terms may be written in the form

$$V = \frac{\sqrt{4\pi}\,F}{\varkappa} \iint d\mathbf{x}\, d\mathbf{x}' \qquad (8)$$

$$\times \sum_{s=1}^{3} \psi^{+}(x)\gamma_\mu\gamma_5\tau_s\psi(x)\,\frac{\partial K(\mathbf{x}-\mathbf{x}')}{\partial x_\mu'}\,\varphi_s(x')$$

where $F = 2fg$, and represents a pseudovector interaction of pions with extended nucleons, exactly as in the strong coupling theory. The function $K(\mathbf{x}-\mathbf{x}')$ now plays the role of form-factor. This factor has the effect that pions and nucleons do not interact at a point but over a region of size (h/Mc). For the theory to be consistent, it is necessary that $(h/Mc) \ll (h/\mu c)$, where μ is the pion mass.

3. CONCLUSIONS

Our proposal of an indirect pion-nucleon interaction thus actually implies the existence of a core of size (h/Mc) within the nucleon. The core is surrounded by a pion shell with size of order $(h/\mu c)$. In addition there appears automatically a very short-range interaction between nucleon cores, acting over a region of size (h/Mc), and a similar short-range interaction between pions. The core itself is described non-relativistically, just like the nucleus in a relativistic theory of the atom.

We intend later to develop the theory in a more consistent way, starting from the Lagrangian method, which will allow a more symmetrical relativistic treatment of the interaction. It would also be interesting to introduce an indirect interaction through K-mesons, renormalized so as to remove completely all the divergences of meson theory. However, it is doubtful whether any indirect interaction exists which would lead to an unrenormalized pseudovector renormalized interaction between pion and nucleon.

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The Quantum Theory of Ferromagnetism

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On the basis of a many-electron quantum-mechanical model of a crystal, a calculation is made of the magnetic interaction of the electrons of a ferromagnetic. The energy spectrum of the system is calculated for the case of low temperatures. Taking account of the magnetic interaction terms in the original Hamiltonian of the system leads to the appearance in the energy both of components of quasi-classical magnetic type and of anisotropic (magnetic) exchange terms. The latter may have considerable importance, for example, in the calculation of relaxation phenomena in ferromagnetics and antiferromagnetics. Also calculated is the energy of the system in the approximation of energetic centers of gravity (the case of high temperatures). An expression is obtained for the free energy as a function of the magnitude of the magnetization and its orientation in the crystal (the magnetic anisotropy energy) for temperatures close to the Curie point.

1. INTRODUCTION

AS is well known (see, for example reference 1, Part III), the ferromagnetic state of matter is determined by electric (exchange) interactions, connected with the dependence of the electric energy on the spin state of the microsystem; this dependence is a consequence of the basic postulates of quantum mechanics. The magnetic interaction between the electrons of ferromagnetic bodies is several orders of magnitude smaller than the electric interaction. Nevertheless, a whole set of important phenomena in ferromagnetics – for example, their purely magnetic properties, their galvanomagnetic properties, relaxation phenomena, and so on – are basically distinguished by magnetic interaction, despite its relative weakness. A specific property of magnetic interaction is its anisotropic (tensorial) character. There exist quite a large variety of types of magnetic interaction: spin-spin interaction between electrons, and also between electrons and ions; spin-orbit interaction between electrons and ions; orbit-orbit interaction between electrons; etc. In a rigorous formulation of the problem, the calculation of magnetic interaction should be performed within the framework of relativistic quantum mechanics. However, the absence of a fully developed relativistic many-electron theory forces us to perform this calculation by the methods of nonrelativistic theory, in which we simply take into account the existence of the electron spin and, using the methods of the correspondence principle, write down the magnetic part of the Hamiltonian operator of a system of electrons.

The energy operator can be decomposed into two parts – an additive part, into which enter the kinetic energy terms and the interaction with external fields, and a binary part, which takes account of interactions of electrons in pairs. In a calculation of the eigenvalues of the energy operator of a system of electrons, along with quasiclassical (or quasinucleonic) terms, which take account of electric and magnetic interactions of "smeared" electrons, there are also obtained terms special to quantum mechanics. Here we can mention in particular the so-called exchange terms, transport energy, excitation energy, etc. It is important to realize that such separation of the energy into components is quite arbitrary; it is merely a result of approximate treatment of the problem of many interacting particles. None the less, in many cases such a separation is entirely permissible.

Until recently, in the treatment of magnetic interaction in problems of the quantum theory of a solid body, no notice has been taken of the fact that the contribution to the energy from this interaction is not exhausted by quasicoulomb terms or by exchange (magnetic) terms alone. In particular, Holstein and Primakoff[2], for example, give a calculation only of the quasiclassical terms in the magnetic part of the energy of a ferromagnetic. In contrast, the work of Tiablikov[3] takes account only of the magnetic exchange interaction tensor and not of the quasiclassical terms in the energy. Meanwhile, in connection with the study of the phenomenon of ferromagnetic resonance and of

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[1] S. V. Vonsovskii, *Contemporary Study in Magnetism*, Moscow, 1953

[2] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940)

[3] S. V. Tiablikov, J. Exper. Theoret. Phys. USSR 20, 661 (1950)

relaxation phenomena in magnetic materials, there has recently arisen an urgent need for a more detailed and systematic investigation of magnetic interaction in these bodies. The present work sets as its aim the laying of a foundation for such an investigation. By starting from the most general possible form of the energy operator of a system of interacting electrons, there is accomplished a separation of the quasi-Coulomb and exchange parts both of the electric and of the magnetic interaction. Specially studied are the case of low temperatures (Section 3), where it is possible to apply the method of elementary excitations (quasiparticles), and the case of high temperatures (Section 4), where it is permissible to apply the method of the energetic center of gravity.

2. THE ENERGY OPERATOR IN THE SECOND-QUANTIZATION REPRESENTATION

We consider a system consisting of N similar, positively charged ions, located at the sites of a crystal lattice. We shall neglect the motion of the ions, treating them as immovable sources of a periodic potential field. In addition, the system contains N electrons, whose motion is taken into account. The energy operator of the system, in coordinate representation, is equal to the sum of an additive part (the kinetic energy and the interaction of the electrons with the external "fields")

$$\sum_{k=1}^{N} \hat{H}_1(\mathbf{q}_k, \nabla_{\mathbf{q}_k}, \hat{\mathbf{s}}_k) = -\frac{\hbar^2}{2m}\sum_{k=1}^{N} \Delta_{\mathbf{q}_k} + \sum_{k,\mathbf{n}=1}^{N} G_{\mathbf{n}}^{(l)}(\mathbf{q}_k) + \sum_{k,\mathbf{n}=1}^{N} G_{\mathbf{n}}^{(m)}(\mathbf{q}_k, \Delta_{\mathbf{q}_k}, \hat{\mathbf{s}}_k) \qquad (2.1)$$

and of a binary part (interaction in pairs)

$$\sum_{k<l=1}^{N} \hat{\Phi}((\mathbf{q}_k - \mathbf{q}_l); \ \nabla_{\mathbf{q}_k}, \nabla_{\mathbf{q}_l}; \ \hat{\mathbf{s}}_k, \hat{\mathbf{s}}_l). \qquad (2.2)$$

Here, $\mathbf{q}_k$ is the radius-vector of the kth electron; $2\pi\hbar$ is Planck's constant; m is the electron mass; $\frac{\hbar}{i}\nabla_{\mathbf{q}_k}$, $-\frac{\hbar}{2m}\Delta_{\mathbf{q}_k}$ and $\hat{\mathbf{s}}_k$ are respectively the momentum, kinetic energy, and spin operators of the kth electron; $G_{\mathbf{n}}^{(l)}(\mathbf{q}_k)$ and $G_{\mathbf{n}}^{(m)}(\mathbf{q}_k, \nabla_{\mathbf{q}_k}, \hat{\mathbf{s}}_k)$ are respectively the electrostatic and the magnetic (spin-orbit) interaction operators of the kth electron and the ion on site $\mathbf{n}$. Thus the energy operator of the system has the form

$$\hat{H} = \sum_{k=1}^{N} \hat{H}_1(\mathbf{q}_k, \nabla_{\mathbf{q}_k}, \hat{\mathbf{s}}_k) + \sum_{k<l=1}^{N} \hat{\Phi}((|\mathbf{q}_k - \mathbf{q}_l|); \ \nabla_{\mathbf{q}_k}, \nabla_{\mathbf{q}_l}; \ \hat{\mathbf{s}}_k, \hat{\mathbf{s}}_l). \qquad (2.3)$$

In the second-quantization representation, (2.3) is written in the form

$$\hat{H} = \sum_{\nu,\nu'} L(\nu, \nu')\,\hat{a}_\nu^+ \hat{a}_{\nu'} + \frac{1}{2}\sum_{\nu_1\nu_2\nu_1'\nu_2'} F(\nu_1, \nu_2; \nu_1', \nu_2')\,\hat{a}_{\nu_1}^+ \hat{a}_{\nu_2}^+ \hat{a}_{\nu_2'} \hat{a}_{\nu_1'}, \qquad (2.4)$$

where

$$L(\nu, \nu') = \int \theta_\nu^*(\mathbf{q}, \mathbf{s})\,\hat{H}_1(\mathbf{q}, \nabla_{\mathbf{q}}, \hat{\mathbf{s}})\,\theta_{\nu'}(\mathbf{q}, \mathbf{s})\,d\mathbf{q}d\mathbf{s}, \qquad (2.5)$$

$$F(\nu_1, \nu_2; \nu_1', \nu_2') = \int \theta_{\nu_1}^*(\mathbf{q}, \mathbf{s})\theta_{\nu_2}^*(\mathbf{q}', \mathbf{s}')\,\hat{\Phi}((\mathbf{q}-\mathbf{q}'); \ \nabla_{\mathbf{q}}, \nabla_{\mathbf{q}'}; \hat{\mathbf{s}}, \hat{\mathbf{s}}')\,\theta_{\nu_1'}(\mathbf{q}, \mathbf{s})\,\theta_{\nu_2'}(\mathbf{q}', \mathbf{s}')\,d\mathbf{q}d\mathbf{q}'d\mathbf{s}d\mathbf{s}' \qquad (2.6)$$

are matrix elements, defined by means of a complete system of orthonormal one-particle functions $\theta_\nu(\mathbf{q}, \mathbf{s})$; the index ν stands for a whole set of quantum numbers, characterizing a given individual state of an electron. $\hat{a}_\nu^+$ and $\hat{a}_\nu$ are the Fermi operators (annihilation and creation operators) of second quantization; they act on functions of the occupation numbers N_ν of individual states. Below, these states will be numbered by means of the radius vector $\mathbf{n}$ of the site near which a given ion is located, together with the spin quantum number $\sigma = \pm 1/2$ ($\nu \equiv \mathbf{n}, \sigma$). The wave functions $\theta_\nu(\mathbf{q}, \mathbf{s})$ have the form of "atomic" (localized) orthonormal functions $\phi(\mathbf{q}_k - \mathbf{R}_\mathbf{n}) = \phi_\mathbf{n}(\mathbf{q}_k)$ ($\mathbf{R}_\mathbf{n}$ is the radius vector of the ion at site $\mathbf{n}$), multiplied* by the spin function $c_\sigma(\mathbf{s}_k)$:

$$\theta_{\mathbf{n},\sigma}(\mathbf{q}, \mathbf{s}) = \varphi_\mathbf{n}(\mathbf{q})\,c_\sigma(\mathbf{s}). \qquad (2.7)$$

Since no consideration will be given below to phenomena connected with transport of electric charge (electric conductivity and the like), it is possible to use a quasihomopolar approximation[4], which is tantamount to the requirement

$$\sum_\sigma \hat{a}_{\mathbf{n}\sigma}^+ \hat{a}_{\mathbf{n}\sigma} = \sum_\sigma \hat{N}_{\mathbf{n}\sigma} = 1 \quad (\text{ for all } \mathbf{n}\ ), \qquad (2.8)$$

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* Here use is made of the smallness of the magnetic interaction in the atom as compared with the electric.

[4] N. N. Bogoliubov, *Lectures on Quantum Statistics*, Kiev, 1949 (in Ukrainian).

and also
$$\Delta = \infty$$

where Δ is the energy of formation of a polar state (the energy gap for formation of an excess-deficiency doublet). The Fermi operators have the properties[4]

$$\hat a^+_{\mathbf{n}\sigma} f(\dots N_{\mathbf{n}\sigma}\dots) = \pm\,\delta(N_{\mathbf{n}\sigma}) f(\dots, N_{\mathbf{n}\sigma}+1, \dots), \tag{2.9}$$
$$\hat a_{\mathbf{n}\sigma} f(\dots N_{\mathbf{n}\sigma}\dots) = \pm\,\delta(1-N_{\mathbf{n}\sigma}) f(\dots, N_{\mathbf{n}\sigma}-1, \dots),$$

where $\delta(x)$ is the Kronecker symbol. By virtue of Eq. (2.9) it is proper to retain in the operator (2.4) only those terms that, acting on functions of the occupation numbers, produce no change in the total number of electrons near each lattice site, as is required by condition (2.8). This gives instead of Eq. (2.4)

$$\hat H = \sum_{\mathbf{n};\sigma,\sigma'} L(\mathbf{n},\sigma;\mathbf{n},\sigma')\hat a^+_{\mathbf{n}\sigma}\hat a_{\mathbf{n}\sigma'} + \frac{1}{2}\sum_{\substack{\mathbf{n}_1\neq\mathbf{n}_2\\(\sigma_1,\sigma_2;\sigma_1',\sigma_2')}} F(\mathbf{n}_1\sigma_1;\mathbf{n}_2\sigma_2;\mathbf{n}_1\sigma_1';\mathbf{n}_2\sigma_2') \tag{2.10}$$
$$\times\,\hat a^+_{\mathbf{n}_1\sigma_1}\hat a^+_{\mathbf{n}_2\sigma_2}\hat a_{\mathbf{n}_2\sigma_2'}\hat a_{\mathbf{n}_1\sigma_1'}$$
$$+\frac{1}{2}\sum_{\substack{\mathbf{n}_1\neq\mathbf{n}_2\\(\sigma_1,\sigma_2;\sigma_1',\sigma_2')}} F(\mathbf{n}_1,\sigma_1;\mathbf{n}_2,\sigma_2;\mathbf{n}_2\sigma_1';\mathbf{n}_1\sigma_2')\hat a^+_{\mathbf{n}_1\sigma_1}\hat a^+_{\mathbf{n}_2\sigma_2}\hat a_{\mathbf{n}_1\sigma_2'}\hat a_{\mathbf{n}_2\sigma_1'}.$$

By virtue of Eqs. (2.1) and (2.2), the matrix elements (2.5) and (2.6) entering into Eq. (2.10) can be put into the form

$$L(\mathbf{n},\sigma;\mathbf{n},\sigma') \tag{2.11}$$
$$=\int\varphi^*_{\mathbf{n}}(\mathbf{q})\left[-\frac{\hbar^2}{2m}\Delta_{\mathbf{q}}+G^{(l)}_{\mathbf{n}}(\mathbf{q})\right]\varphi_{\mathbf{n}}(\mathbf{q})\,d\mathbf{q}\,\delta_{\sigma\sigma'}$$
$$+\int\varphi^*_{\mathbf{n}}(\mathbf{q})\Big[\sum_{\mathbf{n}'(\neq\mathbf{n})}G^{(l)}_{\mathbf{n}'}(\mathbf{q})\Big]\varphi_{\mathbf{n}}(\mathbf{q})\,d\mathbf{q}\,\delta_{\sigma\sigma'}$$
$$+\int\varphi^*_{\mathbf{n}}(\mathbf{q})c^*_{\sigma}(\mathbf{s})$$
$$\times\Big[\sum_{\mathbf{n}'}G^{(m)}_{\mathbf{n}'}(\mathbf{q},\nabla_{\mathbf{q}},\hat{\mathbf{s}})\Big]\varphi_{\mathbf{n}}(\mathbf{q})c_{\sigma'}(\mathbf{s})\,d\mathbf{q}\,d\mathbf{s}$$
$$=(E_0+C)\delta_{\sigma\sigma'}+L^{(m)}(\mathbf{n}\sigma,\mathbf{n}\sigma'),$$

$$F(\mathbf{n}_1\sigma_1,\mathbf{n}_2\sigma_2;\mathbf{n}_1'\sigma_1',\mathbf{n}_2'\sigma_2') \tag{2.12}$$
$$=\int\varphi^*_{\mathbf{n}_1}(\mathbf{q})c^*_{\sigma_1}(\mathbf{s})\varphi^*_{\mathbf{n}_2}(\mathbf{q}')c^*_{\sigma_2}(\mathbf{s}')$$
$$\times\hat\Phi\big((\mathbf{q}-\mathbf{q}');\nabla_{\mathbf{q}},\nabla_{\mathbf{q}'};\hat{\mathbf{s}},\hat{\mathbf{s}}'\big)\varphi_{\mathbf{n}_1'}(\mathbf{q})$$
$$\times\,c_{\sigma_1'}(\mathbf{s})\varphi_{\mathbf{n}_2'}(\mathbf{q}')c_{\sigma_2'}(\mathbf{s}')\,d\mathbf{q}\,d\mathbf{q}'\,d\mathbf{s}\,d\mathbf{s}'.$$

In case the function $\phi_{\mathbf{n}}(\mathbf{q})$ is exactly an atomic function, the quantity E_0 can be regarded as the atomic energy of the given state $\phi_{\mathbf{n}}(\mathbf{q})$, and C can be regarded as a quasiclassical energy of electrostatic interaction of the electron of one of the sites with the ions of all the others. The quantity $L^{(m)}$ represents a perturbation of the neutral atomic level, due to spin-orbit interaction. If the functions $\phi_{\mathbf{n}}(\mathbf{q})$ are not atomic functions, these quantities do not have so simple and graphic a meaning.

By taking account of Eq. (2.11), and also of the commutation relations

$$\hat a^+_f\hat a_{f'}+\hat a_{f'}\hat a^+_f=\delta_{ff'};$$
$$\hat a^+_f\hat a^+_{f'}+\hat a^+_{f'}\hat a^+_f=\hat a_f\hat a_{f'}+\hat a_{f'}\hat a_f=0; \tag{2.13}$$

and of the equations derivable from them[4]

$$\hat a^+_{\mathbf{n}_1\sigma_1}\hat a^+_{\mathbf{n}_2\sigma_2}\hat a_{\mathbf{n}_2\sigma_2'}\hat a_{\mathbf{n}_1\sigma_1'}=\hat a^+_{\mathbf{n}_1\sigma_1}\hat a_{\mathbf{n}_1\sigma_1'}\hat a^+_{\mathbf{n}_2\sigma_2}\hat a_{\mathbf{n}_2\sigma_2'},$$
$$\hat a^+_{\mathbf{n}_1\sigma_1}\hat a^+_{\mathbf{n}_2\sigma_2}\hat a_{\mathbf{n}_1\sigma_2'}\hat a_{\mathbf{n}_2\sigma_1'}=-\hat a^+_{\mathbf{n}_1\sigma_1}\hat a_{\mathbf{n}_1\sigma_2'}\hat a^+_{\mathbf{n}_2\sigma_1}\hat a_{\mathbf{n}_2\sigma_1'},$$

Eq. (2.10) can be written in the form

$$\hat H=N(E_0+C)+\sum_{\mathbf{n};\sigma,\sigma'}L^{(m)}(\mathbf{n},\sigma;\mathbf{n},\sigma')\hat a^+_{\mathbf{n}\sigma}a_{\mathbf{n}\sigma'}$$
$$+\frac{1}{2}\sum_{\substack{\mathbf{n}_1\neq\mathbf{n}_2\\(\sigma,\sigma_2;\sigma_1',\sigma_2')}}[F(\mathbf{n}_1\sigma_1,\mathbf{n}_2\sigma_2;\mathbf{n}_1\sigma_1',\mathbf{n}_2\sigma_2') \tag{2.14}$$
$$-F(\mathbf{n}_1\sigma_1,\mathbf{n}_2\sigma_2;\mathbf{n}_2\sigma_2',\mathbf{n}_1\sigma_1')]\,\hat a^+_{\mathbf{n}_1\sigma_1}\hat a_{\mathbf{n}_1\sigma_1'}\hat a^+_{\mathbf{n}_2\sigma_2}\hat a_{\mathbf{n}_2\sigma_2'}.$$

The matrix elements of the first term in the double sum (over $\mathbf{n}_1$ and $\mathbf{n}_2$) describe the quasiclassical electric and magnetic interaction of the electrons; those of the second term describe the exchange interaction (both electric and magnetic). The older of magnitude of these terms is the same; therefore it is not legitimate to consider them separately, as was done in references 2 and 3. The problem of the present work, as was mentioned above, is the systematic simultaneous calculation of these terms in the energy operator of the system.

3. THE ENERGETIC SPECTRUM OF A SYSTEM IN THE CASE OF MAGNETIC QUASI-SATURATION (THE LOW TEMPERATURE REGION)

The determination of the eigenvalues of the operator (2.14) in the general case leads to insuperable mathematical difficulties. Therefore we must limit ourselves to the investigation of special cases, in which the operator (2.14) can be reduced to diagonal form by separating additive terms in it, corresponding to one or another elementary excitation. The eigenvalues of the energy operator of these elementary excitations (or quasiparticles)[5], having the form of the energy operators of oscillators of some "field", can be written down at once in explicit form, without solving the wave equation.

Such a computation program can be easily carried out if it turns out to be possible to introduce some "neutral" state of the system with respect to the given type of excitation, and to limit oneself thereafter to the consideration only of small departures of the motion of the particles of the system being studied from this "neutral background of motion". In the case being considered, that of a ferromagnetic, it is necessary to take as "neutral" state a state of spin saturation, and the small departures from it will have the form of elementary excitations or quasiparticles, which are called spin waves or ferromagnons. In a first approximation, these ferromagnons can be treated as an ideal gas, obeying symmetric quantum statistics. It must be emphasized that the ferromagnons represent collective degrees of freedom in the motion of a complex system of interacting electrons, located in an ionic crystal lattice. In this treatment no attention is paid to individual degrees of freedom of electrons. The diagonalization of the operator (2.14) is accomplished by application of successive unitary transformation operations. Before starting on this, it is necessary in Eq. (2.14) to perform an explicit summation over spin variables, and also to go over from the Fermi amplitudes $\hat{a}_{\mathbf{n}\sigma}$ to the Bose operator $\hat{\mathbf{b}}_{\mathbf{n}}$ in accordance with the known relations[4]

$$\hat{a}^+_{\mathbf{n},-1/2}\hat{a}_{\mathbf{n},1/2} = (1-\hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}})^{1/2}\hat{b}_{\mathbf{n}}, \quad \hat{a}^+_{\mathbf{n},1/2}\hat{a}_{\mathbf{n},-1/2} = \hat{b}^+_{\mathbf{n}}(1-\hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}})^{1/2},$$
$$\hat{a}^+_{\mathbf{n},-1/2}\hat{a}_{\mathbf{n},-1/2} = 1-\hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}}, \quad \hat{a}^+_{\mathbf{n},1/2}\hat{a}_{\mathbf{n},1/2} = \hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}}. \tag{3.1}$$

The operators $\hat{\mathbf{b}}_{\mathbf{n}}$ satisfy the commutation relation

$$\hat{b}_{\mathbf{n}}\hat{b}^+_{\mathbf{n}'} - \hat{b}^+_{\mathbf{n}'}\hat{b}_{\mathbf{n}} = \delta_{\mathbf{n}\mathbf{n}'}. \tag{3.2}$$

If the system is in a state close to magnetic saturation (low temperatures), then in the operator (2.14), after the substitution (3.1), it is possible to expand the operator functions $(1-\hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}})^{1/2}$ as in power series in $\hat{b}^+_{\mathbf{n}}\hat{b}$** and to keep only terms containing products of no more than four operators $\hat{\mathbf{b}}_{\mathbf{n}}$. Then instead of Eq. (2.14) we get

$$\begin{aligned}\hat{H} = {} & N(E_0 + C) + \sum_{\mathbf{n}}[L^{(m)}(\mathbf{n}^+,\mathbf{n}^+)\hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}} + L^{(m)}(\mathbf{n}^-,\mathbf{n}^-)(1-\hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}}) \\ & + L^{(m)}(\mathbf{n}^+,\mathbf{n}^-)(\hat{b}^+_{\mathbf{n}} - \hat{b}^+_{\mathbf{n}}\hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}}) + L^{(m)}(\mathbf{n}^-,\mathbf{n}^+)(\hat{b}_{\mathbf{n}} - b^+_{\mathbf{n}}\hat{b}_{\mathbf{n}}\hat{b}_{\mathbf{n}})] \\ & + \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-,\mathbf{n}_2^+;\mathbf{n}_1^+,\mathbf{n}_2^+) - F(\mathbf{n}_1^-,\mathbf{n}_2^+;\mathbf{n}_2^+,\mathbf{n}_1^+)]\hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2} \\ & + \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+,\mathbf{n}_2^-;\mathbf{n}_1^+,\mathbf{n}_2^+) - F(\mathbf{n}_1^+,\mathbf{n}_2^-;\mathbf{n}_2^+,\mathbf{n}_1^+)]\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2} \\ & + \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+,\mathbf{n}_2^+;\mathbf{n}_1^-,\mathbf{n}_2^+) - F(\mathbf{n}_1^+,\mathbf{n}_2^+;\mathbf{n}_2^+,\mathbf{n}_1^-)]\hat{b}^+_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2} \\ & + \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+,\mathbf{n}_2^+;\mathbf{n}_1^+,\mathbf{n}_2^-) - F(\mathbf{n}_1^+,\mathbf{n}_2^+;\mathbf{n}_2^-,\mathbf{n}_1^+)]\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2} \\ & + \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_1^+,\mathbf{n}_2^+) - F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_2^+,\mathbf{n}_1^+)]\hat{b}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2} \\ & + \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+,\mathbf{n}_2^+;\mathbf{n}_1^-,\mathbf{n}_2^-) - F(\mathbf{n}_1^+,\mathbf{n}_2^+;\mathbf{n}_2^-,\mathbf{n}_1^-)]\hat{b}^+_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}\end{aligned} \tag{3.3}$$

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[5] S. V. Vonsovskii, *Collection Dedicated to the Memory of S. I. Vavilov*, Publishing House of the Academy of Sciences, USSR, 1952, p. 363

** As is shown by Kubo[6], in this case we have

$$(1-\hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}})^{1/2} = 1-\hat{b}^+_{\mathbf{n}}\hat{b}_{\mathbf{n}}.$$

[6] R. Kubo, Phys. Rev. **87**, 568 (1952)

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^+\mathbf{n}_2^-) - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+)]\,\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}$$

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^-)]\,\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2}$$

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^+) - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^+, \mathbf{n}_1^-)]\,\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2}$$

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^-) - F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^-, \mathbf{n}_1^+)]\,\hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}$$

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-) - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-)](1 - \hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1} - \hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2})$$

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-)$$

$$- F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-)]\,(\hat{b}^+_{\mathbf{n}_1} - \hat{b}^+_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1} - \hat{b}^+_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2})$$

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^-)$$

$$- F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^-, \mathbf{n}_1^-)]\,(\hat{b}^+_{\mathbf{n}_2} - \hat{b}^+_{\mathbf{n}_2}\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2} - \hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2})$$

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-)$$

$$- F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+)]\,(\hat{b}_{\mathbf{n}_1} - \hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1} - \hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2})$$

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^+)$$

$$- F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^+, \mathbf{n}_1^-)]\,(\hat{b}_{\mathbf{n}_2} - \hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2} - \hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2}).$$

Here an abbreviated form of notation has been adopted, for example

$$F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^+)$$

$$= F(\mathbf{n}_1, {}^1/_2; \mathbf{n}_2, -{}^1/_2; \mathbf{n}_1, -{}^1/_2; \mathbf{n}_2, +{}^1/_2)$$

etc.

Among the terms in Eq. (3.3), besides terms quadratic in the Bose operators, there are terms in single operators, and also terms with triple products. It is easy to show that the linear terms give an additive "correction" to the energy of the system. To show this, we perform a unitary transformation from the operators $\hat{b}_{\mathbf{n}}$ and $\hat{b}_{\mathbf{n}}^+$ to their Fourier components by means of the formulas

$$\hat{b}_{\mathbf{n}} = N^{-1/2}\sum_{\nu}\hat{b}_{\mathbf{g}_\nu}\exp\{i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\}, \qquad (3.4)$$

$$\hat{b}_{\mathbf{n}}^+ = N^{-1/2}\sum_{\nu} b^+_{\mathbf{g}_\nu}\exp\{-i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\}.$$

Then for the linear terms in the additive part of Eq. (3.3), we have, for instance,

$$\sum_{\mathbf{n}} L^{(m)}(\mathbf{n}^+, \mathbf{n}^-)\,\hat{b}^+_{\mathbf{n}}$$

$$= N^{-1/2}L^{(m)}(0^+, 0^-)\sum_{\nu}\hat{b}^+_{\mathbf{g}_\nu}\sum_{\mathbf{n}}\exp\{-i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\}$$

$$= N^{-1/2}L^{(m)}(0^+, 0^-)\sum_{\nu}\hat{b}^+_{\mathbf{g}_\nu}\delta(\mathbf{g}_\nu)$$

$$= N^{-1/2}L^{(m)}(0^+, 0^-)\,\hat{b}^+_0.$$

(here the fact has been used that, with neglect of surface effects, the matrix elements $L^{(m)}(\mathbf{n}^+, \mathbf{n}^-)$ are independent of the lattice-site number **n**). Similarly for the terms in the binary part of Eq. (3.3) we have, for instance,

$$\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-)$$

$$- F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+)]\,\hat{b}_{\mathbf{n}_1}$$

$$= \frac{1}{2}\sum_{\mathbf{n}_1, \mathbf{n}_2} f(\mathbf{n}_1 - \mathbf{n}_2)\sum_{\nu}\hat{b}_{\mathbf{g}_\nu}\exp\{i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\};$$

we make the substitution of summation variables $\mathbf{n} = \mathbf{n}_1 - \mathbf{n}_2$, which gives

$$\frac{1}{2}\sum_{\mathbf{n}, \nu} f(\mathbf{n})\,\hat{b}_{\mathbf{g}_\nu}\sum_{\mathbf{n}_1}\exp\{i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\}$$

$$= \frac{1}{2}\sum_{\mathbf{n}, \nu} f(\mathbf{n})\,\hat{b}_{\mathbf{g}_\nu}\delta(\mathbf{g}_\nu) = \frac{1}{2}\sum_{\mathbf{n}} f(\mathbf{n})\,\hat{b}_0.$$

Thus in Eq. (3.3), besides the additive part, there are quadratic and third-degree terms in the operators $\hat{b}$. On collecting similar terms and splitting Eq. (3.3) into two parts, we get

$$\hat{H} = \hat{H}_0 + \hat{H}', \tag{3.5}$$

where

$$\begin{aligned} \hat{H}_0 = U_0 + \sum_{\mathbf{n}} [L^{(m)}(0^+, 0^+) \\ - L^{(m)}(0^-, 0^-)] \hat{b}^+_{\mathbf{n}} \hat{b}_{\mathbf{n}} \\ + \frac{1}{2} \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^+) \\ - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^+, \mathbf{n}_1^+)] \hat{b}_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_2} \\ + \frac{1}{2} \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^-) \\ - F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_2^-, \mathbf{n}_1^-)] \hat{b}^+_{\mathbf{n}_1} \hat{b}^+_{\mathbf{n}_2} \\ + \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-] \\ - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-) \\ + F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-)] \hat{b}^+_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_1} \\ + \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^+) \\ - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^+, \mathbf{n}_1^-)] \hat{b}^+_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_2} + 2\mathfrak{M}\mathscr{H} \sum_{\mathbf{n}} \hat{b}^+_{\mathbf{n}} \hat{b}_{\mathbf{n}}. \end{aligned} \tag{3.6}$$

Also taken into account here is the energy with respect to the external magnetic field H;

$$\mathfrak{M} = e\hbar/2mc, \tag{3.7}$$

U_0 is the additive part of the energy that does not depend on the spin distribution, and

$$\begin{aligned} \hat{H}' = \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [L^{(m)}(0^+, 0^-)\delta_{\mathbf{n}_1\mathbf{n}_2} + F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-) \\ - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-)] \hat{b}^+_{\mathbf{n}_1} \hat{b}^+_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_1} \\ - \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [L^{(m)}(0^-, 0^+)\delta_{\mathbf{n}_1\mathbf{n}_2} + F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-) \\ - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+)] \hat{b}^+_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_2} \hat{b}_{\mathbf{n}_1} \\ + \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^-) - F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_2^-, \mathbf{n}_1^+) \\ - F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^-) \\ + F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^-, \mathbf{n}_1^-)] \hat{b}^+_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_1} \hat{b}^+_{\mathbf{n}_2} \\ + \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^+) \\ - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-) \\ + F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+)] \hat{b}_{\mathbf{n}_1} \hat{b}^+_{\mathbf{n}_2} \hat{b}_{\mathbf{n}_2}. \end{aligned} \tag{3.8}$$

From Eq. (2.12) it is evident that

$$\begin{aligned} F(\mathbf{n}_1\sigma_1, \mathbf{n}_2\sigma_2; \mathbf{n}_1\sigma_1', \mathbf{n}_2\sigma_2') \\ = F[0\sigma_1, (\mathbf{n}_2 - \mathbf{n}_1)\sigma_2; 0\sigma_1', (\mathbf{n}_2 - \mathbf{n}_1)\sigma_2'], \\ F(\mathbf{n}_1\sigma_1, \mathbf{n}_2\sigma_2; \mathbf{n}_2\sigma_2', \mathbf{n}_1\sigma_1') \\ = F[0\sigma_1, (\mathbf{n}_2 - \mathbf{n}_1)\sigma_2; (\mathbf{n}_2 - \mathbf{n}_1)\sigma_2', 0\sigma_1'], \end{aligned}$$

that is, the matrix elements of F depend only on the distance between lattice sites, $\mathbf{R}_{\mathbf{n}_2} - \mathbf{R}_{\mathbf{n}_1}$. From the requirement that the operator (2.2) be Hermitian, it follows further that in Eq. (3.6) the coefficients of $\hat{\mathbf{b}}_{\mathbf{n}_1}\hat{\mathbf{b}}_{\mathbf{n}_2}$ and $\hat{\mathbf{b}}^+_{\mathbf{n}_1}\hat{\mathbf{b}}^+_{\mathbf{n}_2}$, and in Eq. (3.8) the coefficients of $\hat{\mathbf{b}}^+_{\mathbf{n}_1}\hat{\mathbf{b}}_{\mathbf{n}_1}\hat{\mathbf{b}}^+_{\mathbf{n}_2}$ and $\hat{\mathbf{b}}_{\mathbf{n}_1}\hat{\mathbf{b}}^+_{\mathbf{n}_1}\hat{\mathbf{b}}_{\mathbf{n}_2}$, are complex conjugates.

Using these properties of the matrix elements, we perform the Fourier transformation (3.4) in Eqs. (3.6) and (3.8):

$$\begin{aligned} \hat{H}_0 = U_0 + \sum_{\nu} [A_0 + A_1(\mathbf{g}_\nu)] \hat{b}^+_{\mathbf{g}_\nu} \hat{b}_{\mathbf{g}_\nu} \\ + \frac{1}{2} \sum_{\nu} B(\mathbf{g}_\nu) \hat{b}_{-\mathbf{g}_\nu} \hat{b}_{\mathbf{g}_\nu} + \frac{1}{2} \sum_{\nu} B^*(\mathbf{g}_\nu) \hat{b}^+_{-\mathbf{g}_\nu} \hat{b}^+_{\mathbf{g}_\nu}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \hat{H}' = \sum_{\nu,\nu'} [C_0 + C_1(\mathbf{g}_\nu)] \hat{b}^+_{\mathbf{g}_\nu} \hat{b}_{\mathbf{g}_\nu + \mathbf{g}_{\nu'}} \hat{b}^+_{\mathbf{g}_{\nu'}} \\ + \sum_{\nu,\nu'} [C_0^* + C_1^*(\mathbf{g}_\nu)] \hat{b}^+_{\mathbf{g}_\nu} \hat{b}_{\mathbf{g}_\nu - \mathbf{g}_{\nu'}} \hat{b}_{\mathbf{g}_{\nu'}}, \end{aligned} \tag{3.10}$$

where the following symbols have been introduced:

$$\begin{aligned} A_0 = L^{(m)}(0^+, 0^+) - L^{(m)}(0^-, 0^-) \\ + 2\mathfrak{M}\mathscr{H} + \sum_{\mathbf{n}} [F(0^+, \mathbf{n}^-; 0^+, \mathbf{n}^-) \\ - F(0^+, \mathbf{n}^-; \mathbf{n}^-, 0^+) - F(0^-, \mathbf{n}^-; 0^-, \mathbf{n}^-) \\ + F(0^-, \mathbf{n}^-; \mathbf{n}^-, 0^-)]; \end{aligned} \tag{3.11}$$

$$\begin{aligned} A_1(\mathbf{g}_\nu) = \sum_{\mathbf{n}} [F(0^+, \mathbf{n}^-; 0^-, \mathbf{n}^+) \\ - F(0^+, \mathbf{n}^-; \mathbf{n}^+, 0^-)] \exp\{i\mathbf{g}_\nu \mathbf{R}_{\mathbf{n}}\}; \end{aligned}$$

$$B(\mathbf{g}_\nu) = \sum_{\mathbf{n}} [F(0^-, \mathbf{n}^-; 0^+, \mathbf{n}^+)$$
$$- F(0^-, \mathbf{n}^-; \mathbf{n}^+, 0^+)] \exp\{i\mathbf{g}_\nu \mathbf{R}_\mathbf{n}\};$$
$$C_0 = -N^{-1/2} \sum_{\mathbf{n}} [L^{(m)}(0^+, 0^-) + F(0^+, \mathbf{n}^-; 0^-, \mathbf{n}^-)$$
$$- F(0^+, \mathbf{n}^-; \mathbf{n}^-, 0^-)];$$
$$C_1(\mathbf{g}_\nu) = N^{-1/2} \sum_{\mathbf{n}} [F(0^+, \mathbf{n}^+; 0^+, \mathbf{n}^-)$$
$$- F(0^+, \mathbf{n}^+; \mathbf{n}^-, 0^+) - F(0^-, \mathbf{n}^+; 0^-, \mathbf{n}^-)$$
$$+ F(0^-, \mathbf{n}^+; \mathbf{n}^-, 0^-)] \exp\{-i\mathbf{g}_\nu \mathbf{R}_\mathbf{n}\},$$

and $B^*(\mathbf{g})$ and $C^*(\mathbf{g})$ are the complex conjugates of the coefficients $B(\mathbf{g})$ and $C(\mathbf{g})$ respectively. The operator $\hat{H}_0$ describes the energy of elementary excitations of the system near magnetic spin saturation, and the operator $\hat{H}'$ may be regarded as a perturbation energy, which takes account in kinetic phenomena of the "collision" processes between quasiparticles-ferromagnons.

To reduce the operator $\hat{H}_0$ to diagonal form, it is necessary[2,4] to perform the double unitary transformation

$$\hat{b}_\mathbf{g} = 2^{-1/2} e^{i\varphi_\mathbf{g}} (\hat{d}_\mathbf{g} + \hat{d}_{-\mathbf{g}}), \tag{3.12}$$
$$b_{-\mathbf{g}} = 2^{-1/2} e^{i\varphi_{-\mathbf{g}}} (\hat{d}_\mathbf{g} - \hat{d}_{-\mathbf{g}}),$$
$$\hat{d}_\mathbf{g} = l^+ \hat{c}_\mathbf{g} + l^- \hat{c}^+_\mathbf{g}, \quad d_{-\mathbf{g}} = l^+ \hat{c}_{-\mathbf{g}} - l^- \hat{c}^+_{-\mathbf{g}},$$

where

$$l^\pm = \left[\frac{1}{2} \frac{A_\mathbf{g} \pm (A^2_\mathbf{g} - |B(\mathbf{g})|^2)^{1/2}}{(A^2_\mathbf{g} - |B(\mathbf{g})|^2)^{1/2}}\right]^{1/2},$$
$$A_\mathbf{g} = A_0 + A_1(\mathbf{g}). \tag{3.13}$$

As a result of laborious but elementary transformations, with use of the commutation relations (3.2), we find

$$\hat{H}_0 = U_0 + \sum_{\mathbf{g}_\nu} [(A^2_{\mathbf{g}_\nu} - |B(\mathbf{g}_\nu)|^2)^{1/2} \hat{c}^+_{\mathbf{g}_\nu} \hat{c}_{\mathbf{g}_\nu}$$
$$+ \frac{1}{2}(A^2_{\mathbf{g}_\nu} - |B(\mathbf{g}_\nu)|^2)^{1/2} - \frac{1}{2} A_{\mathbf{g}_\nu}], \tag{3.14}$$

where $\hat{c}^+_{\mathbf{g}_\nu} \hat{c}_{\mathbf{g}_\nu} = \hat{N}_{\mathbf{g}_\nu}$ is the ferromagnon number operator in the state with quasimomentum $\mathbf{g}_\nu$; it has the eigenvalues 0,1,2,3,... The expression (3.14) represents the energy levels of the system of interacting electrons in the state of quasisaturation in the first approximation. The operator $\hat{H}'$, as was shown above, can be regarded as a small perturbation; it determines interactions between ferromagnons in the form of collisions between them, leading to a change of the quasimomenta of these quasiparticles or to their creation and annihilation. It is important to note that the result obtained is general for an arbitrary concrete form of the interaction between ferromagnetic electrons. Therefore this result can serve our original purpose, for example, the study of the phenomenon of magnetic anisotropy, of the dependence of spontaneous magnetization on the external field, of relaxation processes, etc.

As an example we consider the case when electric and magnetic spin-spin interaction is present. In this case the binary-interaction operator (2.2) has the form

$$\hat{\Phi}((\mathbf{q} - \mathbf{q}'); \nabla_\mathbf{q}; \nabla'_\mathbf{q}; \hat{\mathbf{s}}, \hat{\mathbf{s}}')$$
$$= V(|\mathbf{q} - \mathbf{q}'|) + \frac{e^2}{m^2c^2} |\mathbf{q} - \mathbf{q}'|^{-5}$$
$$\times [|\mathbf{q} - \mathbf{q}'|^2 (\hat{\mathbf{s}}\hat{\mathbf{s}}') - 3(\hat{\mathbf{s}}(\mathbf{q} - \mathbf{q}'))(\hat{\mathbf{s}}'(\mathbf{q} - \mathbf{q}'))], \tag{3.15}$$

and the matrix elements (2.12) will be expressed by the integrals

$$C_\mathbf{n} = \int V(|\mathbf{q} - \mathbf{q}'|) |\varphi(\mathbf{q})|^2 |\varphi(\mathbf{q}' - \mathbf{R}_\mathbf{n})|^2 \, d\mathbf{q} \, d\mathbf{q}'; \tag{3.16}$$
$$E_\mathbf{n} = \beta \int |\varphi(\mathbf{q})|^2 |\varphi(\mathbf{q}' - \mathbf{R}_\mathbf{n})|^2 |\mathbf{q} - \mathbf{q}'|^{-3} \left[1 - 3 \frac{(\mathbf{q} - \mathbf{q}')^2_z}{|\mathbf{q} - \mathbf{q}'|^2}\right] d\mathbf{q} \, d\mathbf{q}';$$
$$E^{(i,l)}_\mathbf{n} = 3\beta \int |\varphi(\mathbf{q})|^2 |\varphi(\mathbf{q}' - \mathbf{R}_\mathbf{n})|^2 |\mathbf{q} - \mathbf{q}'|^{-5} (\mathbf{q} - \mathbf{q}')_i (\mathbf{q} - \mathbf{q}')_l \, d\mathbf{q} \, d\mathbf{q}';$$
$$I_\mathbf{n} = \int \varphi^*(\mathbf{q}) \varphi^*(\mathbf{q}' - \mathbf{R}_\mathbf{n}) V(|\mathbf{q} - \mathbf{q}'|) \varphi(\mathbf{q}') \varphi(\mathbf{q} - \mathbf{R}_\mathbf{n}) \, d\mathbf{q} \, d\mathbf{q}';$$
$$J_\mathbf{n} = \beta \int \varphi^*(\mathbf{q}) \varphi^*(\mathbf{q}' - \mathbf{R}_\mathbf{n}) |\mathbf{q} - \mathbf{q}'|^{-3} \left[1 - 3 \frac{(\mathbf{q} - \mathbf{q}')^2_z}{|\mathbf{q} - \mathbf{q}'|^2}\right] \varphi(\mathbf{q}') \varphi(\mathbf{q} - \mathbf{R}_\mathbf{n}) \, d\mathbf{q} \, d\mathbf{q}';$$
$$J^{(i,l)}_\mathbf{n} = 3\beta \int \varphi^*(\mathbf{q}) \varphi^*(\mathbf{q}' - \mathbf{R}_\mathbf{n}) |\mathbf{q} - \mathbf{q}'|^{-5} (\mathbf{q} - \mathbf{q}')_i (\mathbf{q} - \mathbf{q}')_l \times$$
$$\times \varphi(\mathbf{q}') \varphi(\mathbf{q} - \mathbf{R}_\mathbf{n}) \, d\mathbf{q} \, d\mathbf{q}'$$
$$(i, l = x, y, z), \quad \beta = e^2\hbar^2 / 4m^2c^2 = \mathfrak{M}^2.$$

Using formulas (2.12), (3.15), (3.11), and (3.16), we get

$$A_{\mathbf{g}} = A'_{\mathbf{g}} + A''_{\mathbf{g}}, \tag{3.17}$$

where

$$A'_{\mathbf{g}} = -3\sum_{\mathbf{n}} E_{\mathbf{n}} + \sum_{\mathbf{n}} E_{\mathbf{n}}(1 - e^{i\mathbf{g}\mathbf{R}_{\mathbf{n}}}) + 2\mathfrak{M}\mathscr{H}; \tag{3.18}$$

$$A''_{\mathbf{g}} = \sum_{\mathbf{n}} I_{\mathbf{n}}(1 - e^{i\mathbf{g}\mathbf{R}_{\mathbf{n}}}) + \sum_{\mathbf{n}} J_{\mathbf{n}}(2 + e^{i\mathbf{g}\mathbf{R}_{\mathbf{n}}}); \tag{3.19}$$

$$B_{\mathbf{g}} = B'_{\mathbf{g}} + B''_{\mathbf{g}};$$

here

$$B'_{\mathbf{g}} = -\sum_{\mathbf{n}} [E^{xx}_{\mathbf{n}} - E^{yy}_{\mathbf{n}} + 2iE^{xy}_{\mathbf{n}}]\, e^{i\mathbf{g}\mathbf{R}_{\mathbf{n}}}; \tag{3.20}$$

$$B''_{\mathbf{g}} = \sum_{\mathbf{n}} [J^{xx}_{\mathbf{n}} - J^{yy}_{\mathbf{n}} + 2iJ^{xy}_{\mathbf{n}}]\, e^{i\mathbf{g}\mathbf{R}_{\mathbf{n}}}; \tag{3.21}$$

$$C_{\mathbf{g}} = C'_{\mathbf{g}} + C''_{\mathbf{g}},$$

where

$$C'_{\mathbf{g}} = -\frac{2}{\sqrt{N}} \sum_{\mathbf{n}} [E^{xz}_{\mathbf{n}} + iE^{yz}_{\mathbf{n}}]\, e^{i\mathbf{g}\mathbf{R}_{\mathbf{n}}},$$

$$C''_{\mathbf{g}} = \frac{2}{\sqrt{N}} \sum_{\mathbf{n}} [J^{xz}_{\mathbf{n}} + iJ^{yz}_{\mathbf{n}}]\, e^{i\mathbf{g}\mathbf{R}_{\mathbf{n}}}. \tag{3.22}$$

Substituting (3.17) and (3.19) in (3.14), we find the energy of the system of ferromagnons with account taken of magnetic spin-spin interaction. In external form, Eq. (3.14) agrees with the corresponding expression in reference 2. However, there is here an essential difference in the form of the quantities $A_{\mathbf{g}}$ and $B_{\mathbf{g}}$. The first terms in these quantities [cf. (3.17) and (3.19)] agree exactly with the coefficients $A_{\mathbf{g}}$ and $B_{\mathbf{g}}$ of the cited work, if one treats these terms quasiclassically, considering that the electrons are "bound" to their own lattice sites. These coefficients are expressed by the integrals $E_{\mathbf{n}}$ and $E^{(i,l)}_{\mathbf{n}}$ in Eq. (3.16), which decrease comparatively slowly with the distance between sites, $R_{\mathbf{n}}$ (Coulomb interaction). By virtue of this last fact, in the calculation of the coefficients $A'_{\mathbf{g}}$ and $B'_{\mathbf{g}}$ the summation in the right members of the first lines of formulas (3.18) and (3.20) must be extended over the whole volume of the crystal (as was done in reference 2). Therefore in the specified case of long-range interactions, it is not permissible to limit oneself to the nearest-neighbor approximation, However, in Eqs. (3.17) and (3.19) there also enter the terms $A''_{\mathbf{g}}$ and $B''_{\mathbf{g}}$, which – with the exception of a term representing the electrostatic exchange interaction, $\sum_{\mathbf{n}} I_{\mathbf{n}}(1 - e^{i\mathbf{g}\mathbf{R}_{\mathbf{n}}})$ – were not taken into account in any way in reference 2. The terms with the integrals $J_{\mathbf{n}}$ and $J^{(i,l)}_{\mathbf{n}}$ represent the anisotropic magnetic exchange interaction of the electrons. They may play an essential role in the consideration, for instance, of the phenomenon of magnetic anisotropy – as is clearly evident from the work of Tiablikov[3]. We mention only that the integrals decrease rapidly with the distance between sites, and therefore in the calculation one can limit oneself to the consideration solely of integrals for nearest-neighbor sites. This fact was utilized in reference 3. However, there no account at all was taken of terms of quasiclassical type, for which the nearest-neighbor approximation is not legitimate.

The nondiagonal part of the energy operator (3.10) offers the possibility, in a more general and complete form than in the work of Akhiezer[7], of studying the process of collision both between ferromagnons and between ferromagnons and phonons; for here also collisions are taken into account that are determined by magnetic interactions of the exchange type (cf. the terms containing $C''_{\mathbf{g}}$). In the series of researches reviewed in reference 8 the hypothesis was advanced that this same magnetic anisotropy exchange interaction (sometimes called pseudodipole interaction) was also basically responsible for relaxation processes in ferromagnetics. As far as we know, the literature contains no deduction of this interaction from the general equations of the problem of interacting electrons, such as has been obtained in the present work.

4. THE ENERGY OF A SYSTEM NEAR THE CURIE TEMPERATURE

In the region of high temperatures the method of quasiparticles, at any rate in the form in which it was applied in Section 3, is no longer suitable for calculation of the energy of the system under consideration – a system of interacting electrons in a ferromagnetic. In this case, however, it is possible to get some approximate information about the energy spectrum of the system by using the method of the energetic center of gravity, i.e., by calculation of mean values of energy. It is this procedure that is essentially involved in the familiar method of the molecular field in the theory of the ferromagnetism.

To find the energetic center of gravity, we separate the diagonal terms in formula (2.14) ($\sigma_1 = \sigma'_1$, $\sigma_2 = \sigma'_2$) and replace the operators

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[7] A. I. Akhiezer, J. Phys. (USSR) 10, 217 (1946)

[8] C. Kittel and E. Abrahams, Revs. Mod. Phys. 25, 233 (1953)

$\hat{a}^{+}_{\mathbf{n}\sigma}\hat{a}_{\mathbf{n}\sigma} = \hat{N}_{\mathbf{n}\sigma}$ by their mean values

$$\langle \hat{a}^{+}_{\mathbf{n},\,{}^1\!/_2}\, \hat{a}_{\mathbf{n},\,{}^1\!/_2} \rangle_{\mathrm{cp}} = \langle \hat{N}_{\mathbf{n},\,{}^1\!/_2} \rangle_{\mathrm{cp}} = N^{-1} \sum_{\mathbf{n}} N_{\mathbf{n},\,{}^1\!/_2} = {}^1\!/_2(1 - m), \tag{4.1}$$

$$\langle \hat{a}^{+}_{\mathbf{n},\,-{}^1\!/_2}\, \hat{a}_{\mathbf{n},\,-{}^1\!/_2} \rangle_{\mathrm{cp}} = \langle \hat{N}_{\mathbf{n},\,-{}^1\!/_2} \rangle_{\mathrm{cp}} \tag{4.2}$$

$$= N^{-1} \sum_{\mathbf{n}} N_{\mathbf{n},\,-{}^1\!/_2} = {}^1\!/_2(1 + m),$$

where

$$m = N^{-1} \sum_{\mathbf{n}} [N_{\mathbf{n},\,-{}^1\!/_2} - N_{\mathbf{n},\,{}^1\!/_2}] \tag{4.3}$$

is the relative magnetization in electronic magnetons $\mathfrak{M}$, corresponding to one atom. Then instead of (2.14) we get for the energy

$$\begin{aligned} E = N(E_0 + C) &+ {}^1\!/_2(1 - m) \sum_{\mathbf{n}} L^{(m)}(\mathbf{n}^+, \mathbf{n}^+) + {}^1\!/_2(1 + m) \sum_{\mathbf{n}} L^{(m)}(\mathbf{n}^-, \mathbf{n}^-) \\ &+ {}^1\!/_8(1 - m)^2 \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^+) - F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^+)] \\ &+ {}^1\!/_8(1 + m)^2 \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-) - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-)] \\ &+ {}^1\!/_8(1 - m^2) \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-) - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+) \\ &\qquad + F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^-)]. \end{aligned} \tag{4.4}$$

At first glance it may appear that there are in (4.4) terms varying linearly with m in seeming violation of the symmetry property. However, this is not so; for the terms linear in m represent the contribution to the mean energy of the system from spin-orbit interaction, which can be written , for instance, in the form[9]

$$\sum_{i=1}^{N} \xi(\mathbf{q}_i)\, \hat{\mathbf{L}}_i\, \hat{\mathbf{s}}_i,$$

and therefore these terms in Eq. (4.4) must depend not simply on the direction of the magnetization **m**, but on the relative orientation of the spin and orbital magnetic moments. If there is an antiferromagnetic distribution of orbital moments, then these terms are equal to zero.

If we consider only the electrostatic and magnetic spin-spin interaction, then, using the notation (3.16), we find

$$\begin{gathered} F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^+) = F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-) = E_{\mathbf{n}_1-\mathbf{n}_2} + C_{\mathbf{n}_1-\mathbf{n}_2}, \\ F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^+) = F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-) = I_{\mathbf{n}_1-\mathbf{n}_2} + J_{\mathbf{n}_1-\mathbf{n}_2}, \\ F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-) = F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^+) = - E_{\mathbf{n}_1-\mathbf{n}_2} + C_{\mathbf{n}_1-\mathbf{n}_2}, \\ F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+) = F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^-) = - J_{\mathbf{n}_1-\mathbf{n}_2}, \\ L^{(m)}(\mathbf{n}^+, \mathbf{n}^+) = L^{(m)}(\mathbf{n}^-, \mathbf{n}^-) = 0. \end{gathered} \tag{4.5}$$

On substituting Eq. (4.5) in Eq. (4.4), we find that

$$E = E^{(0)} - \frac{m^2}{4} \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [I_{\mathbf{n}_1-\mathbf{n}_2} - 2(E_{\mathbf{n}_1-\mathbf{n}_2} - J_{\mathbf{n}_1-\mathbf{n}_2})], \tag{4.6}$$

where

$$E^{(0)} = N(E_0 + C) + \frac{1}{2} \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [C_{\mathbf{n}_1-\mathbf{n}_2} - {}^1\!/_2 I_{\mathbf{n}_1-\mathbf{n}_2}].$$

From formula (4.6) it is evident that the energetic center of gravity varies quadratically with the magnetization m, just as in the case of the usual exchange model, in which no account is taken of the magnetic interaction (cf. for instance, reference 10). But now, instead of electrostatic exchange alone, Eq. (4.6) takes account also of the quasi-Coulomb interaction $E_{\mathbf{n}}$, and the magnetic exchange interaction $J_{\mathbf{n}}$.

After the energetic center of gravity has been found, it is possible to calculate the partition function and the free energy of the crystal, and as a result to determine the temperature dependence of the spontaneous magnetization, and also the magnetic anisotropy constant. Through the magnetic interaction integrals there enters a dependence of the energy on the angles between the crystal axes and the spontaneous magnetization. In the approximation under consideration, the diagonal terms, in the case of cubic lattices, contribute by virtue of the symmetry properties only a constant term, independent of the angles. Therefore it is worthwhile to consider the case of a hexagonal lattice, in which the magnetic anisotropy is a first-order effect.

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[9] E. Condon and G. Shortley, *The Theory of Atomic Spectra*, Cambridge University Press, 1935

[10] S. V. Vonsovskii and Ia. S. Shur, *Ferromagnetism*, State Technical Publishers, Moscow-Leningrad, 1948

In the above formulas, the z axis was selected as the axis of spatial quantization of spins. Here it is worthwhile to choose other axes, related to the principal directions in the crystal. We point axis 1 along the hexagonal axis [0001]; axes 2 and 3 are to lie in the basal plane at an angle of 120° to each other. It is easy to show that in the chosen system of coordinates the expression (4.6) for the energy takes the form

$$E(m, \varphi) = \text{const} - \frac{m^2}{N} I + \frac{2m^2}{N} P(1 - \cos^2 \varphi); \tag{4.7}$$

here ϕ is the angle between the vector $\mathbf{m}$ and the hexagonal axis,

$$I = \sum_{\mathbf{n}'} I_{\mathbf{n}-\mathbf{n}'}; \; P = E_1 - J_1 - \frac{1}{4}(E_2 - J_2), \tag{4.8}$$

$$E_1 = \sum_{\mathbf{n}'} \beta \int \frac{(\mathbf{q}-\mathbf{q}')_1^2}{|\mathbf{q}-\mathbf{q}'|^5} |\varphi_{\mathbf{n}}(\mathbf{q})|^2 |\varphi_{\mathbf{n}'}(\mathbf{q}')|^2 \, d\mathbf{q} \, d\mathbf{q}',$$

$$J_1 = \sum_{\mathbf{n}'} \beta \int \varphi_{\mathbf{n}}^*(\mathbf{q}) \varphi_{\mathbf{n}'}^*(\mathbf{q}') \frac{(\mathbf{q}-\mathbf{q}')_1^2}{|\mathbf{q}-\mathbf{q}'|^5} \varphi_{\mathbf{n}}(\mathbf{q}') \varphi_{\mathbf{n}'}(\mathbf{q}) \, d\mathbf{q} \, d\mathbf{q}',$$

$$E_2 = \sum_{\mathbf{n}'} \beta \int \frac{(\mathbf{q}-\mathbf{q}')_2^2}{|\mathbf{q}-\mathbf{q}'|^5} |\varphi_{\mathbf{n}}(\mathbf{q})|^2 |\varphi_{\mathbf{n}'}(\mathbf{q}')|^2 \, d\mathbf{q} \, d\mathbf{q}' = \sum_{\mathbf{n}'} \beta \int \frac{(\mathbf{q}-\mathbf{q}')_3^2}{|\mathbf{q}-\mathbf{q}'|^5} |\varphi_{\mathbf{n}}(\mathbf{q})|^2 |\varphi_{\mathbf{n}'}(\mathbf{q}')|^2 \, d\mathbf{q} \, d\mathbf{q}'$$

$$= 2 \sum_{\mathbf{n}'} \beta \int \frac{(\mathbf{q}-\mathbf{q}')_2 (\mathbf{q}-\mathbf{q}')_3}{|\mathbf{q}-\mathbf{q}'|^5} |\varphi_{\mathbf{n}}(\mathbf{q})|^2 |\varphi_{\mathbf{n}'}(\mathbf{q}')|^2 \, d\mathbf{q} \, d\mathbf{q}',$$

$$J_2 = \sum_{\mathbf{n}'} \beta \int \varphi_{\mathbf{n}}^*(\mathbf{q}) \varphi_{\mathbf{n}'}^*(\mathbf{q}') \frac{(\mathbf{q}-\mathbf{q}')_2^2}{|\mathbf{q}-\mathbf{q}'|^5} \varphi_{\mathbf{n}}(\mathbf{q}') \varphi_{\mathbf{n}'}(\mathbf{q}) \, d\mathbf{q} \, d\mathbf{q}' = \sum_{\mathbf{n}'} \beta \int \varphi_{\mathbf{n}}^*(\mathbf{q}) \varphi_{\mathbf{n}'}^*(\mathbf{q}') \frac{(\mathbf{q}-\mathbf{q}')_3^2}{|\mathbf{q}-\mathbf{q}'|^5} \varphi_{\mathbf{n}}(\mathbf{q}') \varphi_{\mathbf{n}'}(\mathbf{q}) \, d\mathbf{q} \, d\mathbf{q}'$$

$$= 2 \sum_{\mathbf{n}'} \beta \int \varphi_{\mathbf{n}}^*(\mathbf{q}) \varphi_{\mathbf{n}'}^*(\mathbf{q}') \frac{(\mathbf{q}-\mathbf{q}')_2 (\mathbf{q}-\mathbf{q}')_3}{|\mathbf{q}-\mathbf{q}'|^5} \varphi_{\mathbf{n}}(\mathbf{q}') \varphi_{\mathbf{n}'}(\mathbf{q}) \, d\mathbf{q} \, d\mathbf{q}'$$

Standard calculations of the partition function (cf. § 19, par. 3 of reference 10) give for the free energy of the crystal in the absence of an external field

$$F = -NkT \times \ln \operatorname{ch} \left\{ \frac{1}{2\mathfrak{M} NkT} [(I + 4P) - 6P \sin^2 \varphi] M \right\} + \frac{1}{4N\mathfrak{M}^2} [(I + 4P) - 6P \sin^2 \varphi] M^2, \tag{4.9}$$

where $M = N\mathfrak{M}m$ is the magnetization of the crystal. From Eq. (4.9) it is evident that the free energy of a hexagonal crystal, in the first approximation (with respect to magnetic interaction), is a quadratic function of the sine of the angle between the magnetization and the hexagonal axis, in full accord with thermodynamic theory [cf. for instance, reference 10, formula (38,29)]. From the conditions of thermodynamic equilibrium

$$\partial F / \partial M = 0 \quad \text{and} \quad \partial F / \partial \varphi = 0$$

it is possible to find the well-known dependence of magnetization on temperature and on orientation in the crystal. In the temperature region near the Curie point Θ, by taking advantage of the smallness of the argument of the hyperbolic cosine in the first term on the right in (4.9), we find, after expansion in powers of that argument,

$$F = \text{const} + {}^3/_2 NP \left[\frac{I + 4P}{kT} - 1 \right] m^2 \sin^2 \varphi.$$

Here all terms that do not contain angles have been segregated in "const". Thus the magnetic anisotropy constant of a hexagonal crystal near the Curie point is equal to

$$K = {}^3/_2 NP \left[\frac{I + 4P}{kT} - 1 \right] m^2 \approx {}^3/_2 NP \left(\frac{I}{kT} - 1 \right) m^2. \tag{4.10}$$

The exchange energy is equal in order of magnitude to $k\Theta$, therefore the factor in brackets in Eq. (4.10) is equal in order of magnitude to $(\Theta/T - 1)$, i.e., it is of order of magnitude unity. For some relations between the quantities I and P, the factor $[(I + 4P)/kT - 1]$ can change sign with temperature. At the Curie point of constant K approaches zero because of the factor m^2. The factor $(3/2)\,NPm^2$ is of the order of magnitude 10^6 to 10^7 erg cm^{-3} (since $m \sim 1$, $N \sim 10^{22}$cm^{-3}, $P \sim 10^{-6}$erg); this should also be the magnitude of the magnetic anisotropy constant, for instance, in a hexagonal monocrystal of cobalt at room temperature. More detailed quantitative calculations, in this energetic center of gravity approximation, would hardly be worthwhile in view of the roughness of the approximation.

5. CONCLUSIONS

1. A systematic general scheme has been developed for the quantum-mechanical many-electron treatment of magnetic interaction in crystals; it can serve as a point of departure for the study of all effects in ferromagnetic and antiferromagnetic crystals that are determined by this type of interaction between electrons.

2. The case of low temperatures has been specially investigated; here it is possible to use the method of quasiparticles. The diagonal part of the energy operator of the system has been determined and its eigenvalues found. An expression has been found for the energy of the perturbation that takes account of collision processes between ferromagnons.

3. The case of high temperatures has been considered in the energetic center-of-gravity approximation, and some of its concrete applications in the theory of ferromagnetism have been given.

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The Asymptotic Green's Function of Nucleon and Meson in Pseudo-Scalar Theory with Weak Interaction

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The asymptotic Green's function (for $|p^2| \gg m^2$) of nucleon and meson is examined in pseudoscalar theory with pseudoscalar coupling for small values of the coupling constant. The starting point is a set of equations proposed in reference 1. In contradistinction to this work, the renormalization of mass and charge are performed according to a method developed in reference 2, and it is proved that that method removes all infinities from the problem.

I. FORMULATION OF THE EQUATIONS

In reference 2, covariant equations were obtained which described a single nucleon interacting with a pseudoscalar meson field, and the renormalization of mass and charge was performed in them. In the present work, we apply that method of mass and charge renormalization in a problem of some methodological interest, that of the asymptotic behavior (for $|p^2| \gg m^2$) of the Green's functions $G(\mathbf{p})$ of the nucleon and $D(p^2)$ of the meson* for small g^2. In this case, it will be seen that the factor Z_1 (remaining in the equations of reference 2), which can be infinite, is automatically excluded and does not appear in the result. Thus, the method of removing infinities whose definition does not involve perturbation theory will be illustrated on a concrete example.

For the study of the asymptotic Green's function it is necessary to select the dominant terms from the infinite system of "branching" equations. From perturbation theory, it is known that the diagrams for $G(\mathbf{p})$ and $D(p^2)$ behave as $g^{2n} \times [\ln(p^2/m^2)]^m$ where $m \leq n$ [3,4]. For $g^2 \ll 1$, diagrams with $n = m$ dominate. As was proved in reference 1, the condition $n = m$ makes it possible to reduce the infinite system of equations to three equations for the three functions $G(\mathbf{p})$, $D(p^2)$ and $\Gamma_5(\mathbf{p}, \mathbf{p} - \mathbf{k}; -\mathbf{k})$. The function $\Gamma_5(\mathbf{p}, \mathbf{p} - \mathbf{k}; -\mathbf{k})$ actually depends on two momenta (for example, on the initial momentum $\mathbf{p}$ and final momentum $\mathbf{p} - \mathbf{k}$ of the nucleon). For convenience in the following calculation, we write three momenta, the redundant meson momentum being separated by a semi-colon.

In pseudoscalar symmetrical theory, these functions are subject to the following system of equations[1]:

$$\Big\{\mathbf{p} - m_0 - 3\frac{g_0^2}{4\pi^3 i}\gamma_5 \int d^4k\, G(\mathbf{p}-\mathbf{k}) \times \Gamma_5(\mathbf{p}-\mathbf{k}, \mathbf{p}; \mathbf{k})\, D(k^2)\Big\}\, G(\mathbf{p}) = 1; \tag{1}$$

$$\Big\{k^2 - \mu_0^2 + 2\frac{g_0^2}{4\pi^3 i}\int d^4p\, \mathrm{Sp}\, \gamma_5 G(\mathbf{p}) \times \Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k})\, G(\mathbf{p}+\mathbf{k})\Big\}\, D(k^2) = 1; \tag{2}$$

$$\begin{aligned}\Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k}) &= \gamma_5 - \frac{g_0^2}{4\pi^3 i}\int \Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{q}; -\mathbf{q})\, G(\mathbf{p}-\mathbf{q}) \\ &\times \Gamma_5(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}-\mathbf{k}; -\mathbf{k})\, G(\mathbf{p}-\mathbf{q}-\mathbf{k}) \\ &\times \Gamma_5(\mathbf{p}-\mathbf{q}-\mathbf{k}, \mathbf{p}-\mathbf{k}; \mathbf{q})\, D(q^2)\, d^4q.\end{aligned} \tag{3}$$

We clarify the correspondence between these equations and the system of branching equations of reference 2. Equations (1) and (2) coincide* with Eqs. (5) II and (7) II if one substitutes there the expressions for $M_0(\mathbf{p})$ and $P_0(k^2)$ from Eqs. (9) II and (10), II, respectively, and notes that $\Gamma_5 = \gamma_5 + M_1(\mathbf{p}, \mathbf{k})$. If in Eqs. (5) – (10) II, we set $M_n = 0$ for $n \geq 2$ and $P_n = 0$ for $n \geq 1$, then from

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* In reference 2 these functions were denoted by $G_0(p)$, $D_0(p^2)$; the remaining notation agrees with that paper. References to the formulas of reference 2 will be denoted by II.

[1] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 95, 497, 773, 1177 (1954); 96, 261 (1954)

[2] A. D. Galanin, B. L. Ioffe and I. Ia. Pomeranchuk, Dokl. Akad. Nauk SSSR 98, 361 (1954)

[3] A. I. Akhiezer and V. B. Beresetskii, *Quantum Electrodynamics*, Moscow, 1953, p. 210

[4] A. D. Galanin, J. Exper. Theoret. Phys. USSR 27, 417 (1954)

* Symmetrical pseudoscalar theory differs from neutral pseudoscalar theory in the appearance in Eq. (1) of the coefficient 3 in front of the integral, and the appearance in Eqs. (2) and (3) of the corresponding coefficients 2 and – 1.

Eqs. (6) II and (9) II we obtain an equation differing somewhat from Eq. (3):

$$\Gamma_5(\mathbf{p},\ \mathbf{p}-\mathbf{k};\ -\mathbf{k}) = \gamma_5 - \frac{g_0^2}{4\pi^3 i}\gamma_5 \int G(\mathbf{p}-\mathbf{q})\Gamma_5(\mathbf{p}-\mathbf{q},\ \mathbf{p}-\mathbf{q}-\mathbf{k};\ -\mathbf{k}) \times G(\mathbf{p}-\mathbf{q}-\mathbf{k})\Gamma_5(\mathbf{p}-\mathbf{q}-\mathbf{k},\ \mathbf{p}-\mathbf{k};\ \mathbf{q})D(q^2)\,d^4q. \tag{4}$$

If we solve the system of Eqs. (1), (2) and (4) by iteration, then we find that not all diagrams are obtained of the class which interests us. Namely, diagrams will be absent which are obtained by inserting a vertex part in the left-most vertex of the basic diagram given in Fig. 1 for Γ_5 (see, for example, the diagram depicted in Fig. 2).

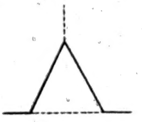
FIG. 1

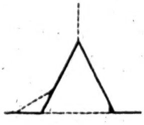
FIG. 2

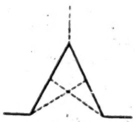
FIG. 3

Now, each diagram in which every integration over an interior meson line leads to an infinity in the unrenormalized theory will belong to the class which interests us. In fact, after renormalization, each infinity is replaced by $\ln(p^2/m^2)$ and asymptotically such a diagram will behave as $g^{2n}[\ln(p^2/m^2)]^n$, i.e., such that we ought to take account of it. In particular, the diagram depicted in Fig. 2 must properly be taken into account, since, asymptotically, it has the form

$$g^4\left(\ln\frac{p^2}{m^2}\right)^2.$$

Therefore, in Eq. (9) II for M_n with $n \geq 2$, one cannot set $M_n = 0$, but rather must take into account the set of those terms which correspond to the insertion of a vertex part in the left-most vertex of Fig. 1. Here it is necessary to select only terms in which each integration over an internal meson line leads to infinities, and not to take into account "overlapping" diagrams [for example, look at Fig. 3 which behaves asymptotically as $g^4\ln(p^2/m^2)$]. Proceeding in the indicated fashion, we pass from Eq. (4) to Eq. (3).

We now perform the renormalization of mass and charge in Eqs. (1) – (3). Since Eqs. (1) and (2) coincide with the corresponding equations of II, their renormalization can be performed just as in II. Therefore, it remains only to renormalize Eq. (3). Following the method developed in II, we introduce in place of $\Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k})$, the renormalized Green's function $\Gamma_5^*(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k}) = Z_1\Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k})$ where $Z_1^{-1} = \gamma_5\Gamma_5(m, m, \mu)$ and m and μ are the experimental masses of nucleon and meson, respectively. We use the following connection between the renormalized and unrenormalized Green's functions of nucleon and meson [compare Eq. (22) II] $G^* = Z_2^{-1}G$, $D^* = Z_3^{-1}D$, where

$$Z_2 = 1 + M_0'^*(m);\ Z_3 = 1 + P_0'^*(\mu^2), \tag{5}$$

and the experimental and fictive charge, g and g_0, respectively, are related by: $g_0^2 = Z_1^2 Z_2^{-2} Z_3^{-1} g^2$. Then, in place of Eq. (3), we get the following equation for the renormalization function $\Gamma_5^*(\mathbf{p}, \mathbf{p}-\mathbf{k}; -\mathbf{k})$:

$$\Gamma_5^*(\mathbf{p},\ \mathbf{p}-\mathbf{k};\ -\mathbf{k}) = Z_1\gamma_5 - \frac{g^2}{4\pi^3 i}\int \Gamma_5^*(\mathbf{p},\ \mathbf{p}-\mathbf{q};\ -\mathbf{q})G^*(\mathbf{p}-\mathbf{q}) \times \Gamma_5^*(\mathbf{p}-\mathbf{q},\ \mathbf{p}-\mathbf{q}-\mathbf{k};-\mathbf{k})G^*(\mathbf{p}-\mathbf{q}-\mathbf{k}) \times \Gamma_5^*(\mathbf{p}-\mathbf{q}-\mathbf{k},\ \mathbf{p}-\mathbf{k};\ \mathbf{q})D^*(q^2)\,d^4q. \tag{6}$$

Equation (6) can be written conveniently in another form, if Z_1 is expressed in terms of the renormalized function and experimental charge:

$$Z_1^{-1} = 1 + \gamma_5 M_1(m, \mu) = 1 + \gamma_5 Z_1^{-1} M_1^*(m, \mu), \tag{7}$$

because $M_1^* = Z_1 M_1$. Thus,

$$Z_1 = 1 - \gamma_5 M_1^*(m, \mu) \tag{7'}$$

and Eq. (6) takes the following form:

$$\Gamma_5^*(\mathbf{p}, \mathbf{p} - \mathbf{k}; -\mathbf{k}) = \gamma_5 + M_1^*(\mathbf{p}, \mathbf{k}) - M_1^*(m, \mu), \tag{8}$$

where

$$M_1^*(\mathbf{p}, \mathbf{k}) = -\frac{g^2}{4\pi^3 i}\int \Gamma_5^*(\mathbf{p}, \mathbf{p} - \mathbf{q}; -\mathbf{q}) \times G^*(\mathbf{p} - \mathbf{q})\Gamma_5^*(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} - \mathbf{k}; -\mathbf{k}) \times G^*(\mathbf{p} - \mathbf{q} - \mathbf{k}) \times \Gamma_5^*(\mathbf{p} - \mathbf{q} - \mathbf{k}, \mathbf{p} - \mathbf{k}; \mathbf{q}) D^*(q^2)\, d^4q. \tag{9}$$

Equations (8) and (9), together with the renormalized Eq. (23) II for $G^*(\mathbf{p})$ and $D^*(k^2)$:

$$\{\mathbf{p} - m - [M_0^*(\mathbf{p}) - M_0(m) - (\mathbf{p} - m) M_0^{*\prime}(m)]\} G^*(\mathbf{p}) = 1; \tag{10}$$

$$M_0^*(\mathbf{p}) = 3\frac{g^2}{4\pi^3 i} Z_1\gamma_5 \int G^*(\mathbf{p} - \mathbf{k})\Gamma_5^*(\mathbf{p} - \mathbf{k}, \mathbf{p}; \mathbf{k}) D(k^2)\, d^4k; \tag{11}$$

$$\{k^2 - \mu^2 - [P_0^*(k^2) - P_0^*(\mu^2) - (k^2 - \mu^2) P_0^{\prime *}(\mu^2)]\} D(k^2) = 1; \tag{12}$$

$$P_0^*(k^2) = -2\frac{g^2}{4\pi^3 i} Z_1 \operatorname{Sp}\gamma_5 \int G^*(\mathbf{p})\Gamma_5^*(\mathbf{p}, \mathbf{p} + \mathbf{k}; \mathbf{k}) G(\mathbf{p} + \mathbf{k})\, d^4p \tag{13}$$

and Eq. (7′) for Z_1 constitute the complete system for the determination of the asymptotic behavior of the functions G^* and D^*.

2. EQUATIONS FOR THE GREEN'S FUNCTION OF THE NUCLEON

Since we are interested only in the asymptotic function G, D and Γ_5[₽], we can neglect m^2 and k^2. Therefore, a solution of Eqs. (7) - (13) will be of the form[₽₽]

$$G(\mathbf{p}) = \mathbf{p}^{-1}F(p^2), \quad D(k^2) = k^{-2}\varphi(k^2), \tag{14}$$

where F and ϕ are slowly varying (for example, logarithmic) functions of their arguments.

The general form of the function $\Gamma_5(\mathbf{p}, \mathbf{p} - \mathbf{k}; -\mathbf{k})$, when its arguments are large compared to m and $|k^2| \gg |p^2|$, is as follows:

$$\Gamma_5(\mathbf{p} - \mathbf{k}, \mathbf{p}; \mathbf{k}) = \gamma_5\Big\{s_0[(\mathbf{p} - \mathbf{k})^2, p^2, k^2] - \frac{\mathbf{k}\mathbf{p}}{k^2 + p^2} s_1[(\mathbf{p} - \mathbf{k})^2, p^2, k^2]\Big\}. \tag{15}$$

The function $s_0[(\mathbf{p} - \mathbf{k})^2, p^2, k^2]$ is a slowly varying function of its arguments when written in terms of the independent variables (this form is suggested by perturbation theory, in which it is easy to write down s_0 in first approximation and verify the following argument).

For $|k^2| \gg |p^2|$, the second term of Eq. (15) is very much smaller than the first; however, it is necessary to take it into account, since, when it is substituted in the mass operator (11), we still get a logarithmically divergent integral. This is connected with the fact that s_1 (as can be shown in perturbation theory in corroboration of the following computation), considered as a function of two variables $s_1 = s_1(p^2, k^2)$, grows as $\ln(k^2/p^2)$ for $|k^2| \gg |p^2|$ but for $|k^2| \sim |p^2|$ it approaches a constant, which we shall neglect.

At first glance, it might seem that momenta k^2 of order p^2 should be significant in the integration over four dimensional k-space which occurs in the renormalized mass operator appearing in Eq. (10). That is correct, however, only for the first term, $\gamma_5 s_0$, of Γ_5 appearing in Eq. (15). Indeed, from perturbation theory it is known that the subtraction from $M_0(\mathbf{p})$ of the expressions $M_0(m)$ and $(\mathbf{p} - m) M_0'(m)$ does not eliminate all infinities. Namely, there occur the so-called "b-diverg-

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[₽] Here and in the following G, D, Γ_5 will denote the renormalized functions and the index* will be omitted.

[₽₽] The method of solving Eqs. (10) - (13) is essentially borrowed from reference 1, where such equations were solved for the case of quantum electrodynamics but without carrying out the renormalization of the equations themselves before solving them. In spite of the similarity of the calculation, we shall carry them out in some detail, in order to illustrate more clearly the singularities which arise in working with the renormalized equations. The results of the present work are obtained also in reference 5 which was based on the method of reference 1,

[5] A. A. Abrikosov, A. D. Galanin and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **97**, 793 (1954)

ences"[6,7] whose removal is effected in perturbation theory by computing the infinite part of Γ_5, and introducing it in turn into the left and right vertices of the diagram for the mass operator. In our method of renormalization this infinity is compensated by the factor Z_1 appearing in Eq. (11). That just the function $s_1(p^2, k^2)$ is responsible for "*b*-divergences" can be checked, for example, in the first approximation of perturbation theory.

Following reference 1, we shall consider the vector p_μ in Eq. (10) as space-like $(p_4^2 - \vec{p}^2) < 0$. Then displaced poles[6] will be absent and the integration on k_4 can be carried out along the imaginary axis $(k_4 \to ik_4)$. Such a transformation we shall call transition to "euclidian metric". First, we consider those parts of the mass operator which arise from the substitution of the function $\gamma_5 s_0$ for Γ_5. These will be denoted by the index $^{(0)}$. Since renormalization makes the integrals over k convergent, one can regard all integrals as carried out up to a limit of order of magnitude p. Since, furthermore, the result will only be logarithmically dependent on the upper limit, its precise value is not essential. Taking into account Eqs. (14) and (15), we get for $M_0^{(0)}(\mathbf{p})$ the following integral

$$M_0^{(0)}(\mathbf{p}) = \tag{16}$$

$$-3\,\frac{g^2}{4\pi^3 i} Z_1 \mathbf{p} \int \frac{1-(\mathbf{pk}/p^2)}{(p-k)^2 k^2}$$

$$\times\; F(k^2)\, s_0(k^2)\, \varphi(k^2)\, d^4k$$

[We exploit the slow variation of the functions F and s_0 replace their arguments $(p-k)^2$ with k^2.] The general form of the mass operator is thus:

$$M_0(\mathbf{p}) = \mathbf{p} f_1(p^2) + m f_2(p^2),$$

where f_1 and f_2 depend only on p^2. It is evident hat in the renormalized expression $\mathbf{p} M_0'(m)$, the essential part for large $|p^2|$ is $\mathbf{p} f_1(m^2)$. Thus, $\mathbf{p} M_0^{(0)'}(m)$ is determined from the same Eq. (16), but under the integral sign, p^2 is set equal to m^2

The transition to euclidian metric and the introduction of a spherical system of coordinates,

$$d^4k = 4\pi k^3 \sin^2\alpha\, dk\, d\alpha, \tag{17}$$

yields (the lower limit is of order m^2)

$$M_0^{(0)}(\mathbf{p}) = \tag{18}$$

$$-\frac{3g^2}{2\pi^2} Z_1 \frac{\mathbf{p}}{|p^2|} \int_{m^2}^{|p^2|} dk^2 F(k^2)\, s_0(k^2)\, \varphi(k^2)$$

$$\times \int_0^\pi \frac{1-(k/|p|)}{1+(k^2/|p^2|)-(2k/|p|)\cos\alpha} \sin^2\alpha\, d\alpha.$$

The integral over the angle α is equal to $(1/4)\pi(2 - k^2/|p^2|)$, if $|k^2| < |p^2|$, and $(1/4)\pi|p^2|/k^2$ if $|k^2| > |p^2|$.

Consequently, we get

$$M_0^{(0)}(\mathbf{p}) =$$

$$-\frac{3g^2}{8\pi} Z_1 \frac{\mathbf{p}}{|p^2|} \int_{m^2}^{|p^2|} dk^2 F(k^2)\, s_0(k^2)\, \varphi(k^2) \left(2 - \frac{k^2}{|p^2|}\right);$$

$$\mathbf{p} M_0^{(0)'}(m) = -\frac{3g^2}{8\pi} Z_1 \mathbf{p} \int_{m^2}^{|p^2|} \frac{dk^2}{k^2} F(k^2)\, s_0(k^2)\, \varphi(k^2).$$

From these formulas, it is seen that the main term is $\mathbf{p} M_0^{(0)'}(m)$. Transforming to the logarithmic variable $z = \ln(k^2/m^2)$ and writing $\xi = \ln(-p^2/m^2)$, $\lambda = g^2/4\pi$, we have

$$M_0^{(0)}(\mathbf{p}) - M_0^{(0)}(m) - (\mathbf{p} - m) M_0^{(0)'}(m) \tag{19}$$

$$= \frac{3}{2} \lambda Z_1 \mathbf{p} \int_0^\xi F(z)\, s_0(z)\, \varphi(z)\, dz.$$

We pass now to the consideration of the second part of the mass operator, which is obtained from the term Γ_5 containing s_1. We shall denote that part of $M_0(\mathbf{p})$ by the index $^{(1)}$. We get evidently

$$M_0^{(1)}(\mathbf{p}) = -\frac{3g^2}{4\pi^3 i} Z_1 \gamma_5 \int (\mathbf{p} - \mathbf{k})^{-1} \gamma_5 \frac{\mathbf{kp}}{k^2} \frac{1}{k^2} \tag{20}$$

$$\times\; s_1(p^2, k^2)\, F(k^2)\, \varphi(k^2)\, d^4k,$$

where, as has already been proved, the essential region of tntegration lies where $|k| \gg |p|$. We can therefore neglect $\mathbf{p}$ as compared with $\mathbf{k}$ in the factor $(\mathbf{p} - \mathbf{k})^{-1}$ under the integral sign but take p as the lower limit of the integral. The result depends logarithmically on this lower limit. After a transition to euclidian metric, and integration over angle, we get

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[6] F. J. Dyson, Phys. Rev. **75**, 1736 (1949)

[7] Al Salam, Phys. Rev. **82**, 217 (1951)

$$M_0^{(1)}(\mathbf{p}) = -3\lambda Z_1 \mathbf{p} \int_{\xi}^{\infty} F(z)\varphi(z)s_1(\xi, z)\,dz. \quad (21)$$

Evidently, the renormalization term is

$$(\mathbf{p} - m)\, M_0^{(1)'}(m) = \quad (22)$$

$$-3\lambda Z_1 \mathbf{p} \int_{0}^{\infty} F(z)\varphi(z)s_1(0, z)\,dz.$$

Substituting Eqs. (18), (21) and (22) into Eq. (10), we can write down a one-dimensional integral equation for the Green's function of the nucleon (after a factor **p** has been removed):

$$1 + \frac{3}{2}\lambda Z_1 \int_{\xi}^{\infty} F(z)\varphi(z)[s_0(z) + 2s_1(\xi, z)]\,dz \quad (23)$$

$$-\frac{3}{2}\lambda Z_1 \int_{0}^{\infty} F(z)\varphi(z)[s_0(z)$$

$$+ 2s_1(0, z)]\,dz = \frac{1}{F(\xi)}.$$

We turn to the construction of a one-dimensional integral equation for Γ_5, and consider, first of all, an equation for the function $s_0(p^2)$. The situation here is analogous to that occurring in the consideration of $M_0^{(0)}(\mathbf{p})$: the integral, renormalized by the subtraction of $M_1(m, \mu)$, converges for $|q| \sim |p|$ (or for $|q|$ of order of magnitude $|k|$, since $|p|$ and $|k|$ have the same order of magnitude in those Γ_5 which are necessary for the computation of the mass operator). Therefore, just as for $M_0^{(0)}(\mathbf{p})$, the main term is $M_1(m, \mu)$, which contains one more power of $\ln(p^2/m^2)$ than $M_1(\mathbf{p}, \mathbf{k})$. Considering the equation for s_0, one can neglect the small additions to Γ_5 which are proportional to s_1 since they would lead to an integral not of logarithmic form, and the result would have at worst one power of $\ln(p^2/m^2)$ less than the main term. Thus, the equation for s_0 has the form

$$s_0(p^2) = 1 + \frac{g^2}{4\pi^3 i}\int (\mathbf{p} - \mathbf{q})^{-1} \quad (24)$$

$$\times \gamma_5(\mathbf{p} - \mathbf{q} - \mathbf{k})^{-1}\gamma_5 q^{-2} s_0^3(q^2)F^2(q^2)\varphi(q^2)\,d^4q \Big|_{\substack{p=m\\ k^2=\mu^2}},$$

where the integration must be performed for $|q| \sim |p|$. Making the transition to the one-dimensional equation by the same method which was used to obtain the Green's function for the nucleon we get

$$s_0(\xi) = 1 - \lambda \int_{0}^{\xi} s_0^3(z)F^2(z)\varphi(z)\,dz. \quad (25)$$

In passing, we remark that Z_1 as calculated from Eq. (7) differs from Eq. (25) only in that integration over z runs to infinity so that

$$Z_1 = s_0(\infty) = 1 - \lambda \int_{0}^{\infty} s_0^3(z)F^2(z)\varphi(z)\,dz. \quad (26)$$

We consider now the equation for the function s_1. We will be interested in those momenta which are essential for the substitution of s_1 in the equation for G. We write down the integral (9) again, indicating all momenata explicitly

$$M_1(\mathbf{p} - \mathbf{k}, \mathbf{p}; \mathbf{k}) = \quad (27)$$

$$-\frac{g^2}{4\pi^3 i}\int \Gamma_5(\mathbf{p} - \mathbf{k}, \mathbf{p} - \mathbf{k} + \mathbf{q}; \mathbf{q})\,G(\mathbf{p} - \mathbf{k} + \mathbf{q})$$

$$\times \Gamma_5(\mathbf{p} - \mathbf{k} + \mathbf{q}, \mathbf{p} + \mathbf{q}; \mathbf{k})$$

$$\times G(\mathbf{p} + \mathbf{q})\,\Gamma_5(\mathbf{p} + \mathbf{q}, \mathbf{p}; -\mathbf{q})\,D(q^2)\,d^4q.$$

As was proved above [see Eq. (20)], the essential domain of momentum is $|k| \gg |p|$. On the other hand, the function s_1 approaches zero if the nucleon possesses momentum of that order of magnitude (below it is proved that this assertion is justified). Therefore, in the Γ_5 on the left in Eq. (27) the function s_1 need not be taken into account. In the following Γ_5, which stands in the middle, the nucleon momentum is in general of a different order of magnitude. However, if the part of Γ_5 containing the function s_1,

$$\frac{\mathbf{k}(\mathbf{p} + \mathbf{q})}{k^2}\, s_1[(p + q)^2,\ k^2],$$

is written out, then one can convince oneself that the same large momenta $|q| \sim |k|$ will be essential in that integral. But then in the function $s_1[(p + q)^2, k^2]$ the arguments are of the order of magnitude unity and it approaches zero. Thus, it is necessary to keep the function s_1 only in the Γ_5 to the right. Substituting Eq. (14), and bearing in mind the equality:

$$\gamma_5(\mathbf{p} - \mathbf{k} + \mathbf{q})^{-1}\gamma_5(\mathbf{p} + \mathbf{q})^{-1}\gamma_5 =$$

$$-\gamma_5(\mathbf{p} + \mathbf{q} - \mathbf{k})^{-2}\left[1 - \frac{\mathbf{k}(\mathbf{p} + \mathbf{q})}{(p + q)^2}\right],$$

we get

$$M_1(\mathbf{p}-\mathbf{k},\mathbf{p};\mathbf{k}) \tag{28}$$
$$=\frac{g^2}{4\pi^3 i}\gamma_5\int(\mathbf{p}+\mathbf{q}-\mathbf{k})^{-1}\left[1-\frac{\mathbf{k}(\mathbf{p}+\mathbf{q})}{(p+q)^2}\right]q^{-2}$$
$$\times\left[s_0+\frac{\mathbf{qp}}{q^2}s_1(p^2,q^2)\right]s_0\cdot s_0F[(p-k+q)^2]$$
$$\times\ F[(p+q)^2]\varphi(q^2)\,d^4q.$$

Now the terms porportional to **kp** which determine s_1 are easily separated. We get

$$\frac{\mathbf{kp}}{k^2}s_1(p^2,k^2)=\frac{g^2}{4\pi^3 i}\int\frac{d^4q}{q^2}(\mathbf{p}-\mathbf{k}+\mathbf{q})^{-2}\left[\frac{\mathbf{k}(\mathbf{p}+\mathbf{q})}{(p+q)^2}s_0\right.$$
$$\left.+\frac{\mathbf{kp}}{(p+q)^2}s_1(p^2,q^2)\right]$$
$$\times s_0\cdot s_0F[(p-k+q)^2]F[(p+q)^2]\varphi(q^2). \tag{29}$$

The integral over q converges, but, if $|k|\gg|p|$, for $|p|\ll|q|\ll|k|$ the integrand will have the form dq/q, and consequently, the integral will increase logarithmically with $|k^2|$, i.e., the function s_1 contains $\ln(k^2/p^2)$. If $|k|\sim|p|$ then the logarithmic growth of the integral does not take place and s_1 is small. This result confirms the earlier conclusion about the form of the function s_1.

In Eq. (29) the arguments of the functions s_0 are not indicated. Since the essential domain for the integration occurs for $|p|\ll|q|\ll|k|$ the argument of the first function s_0 is q^2, but the arguments of the other two functions s_0 are equal to k^2. The arguments of the functions F can be taken: one equal to k^2, the other equal to q^2. Because the integral determining the function s_1 converges, it is not necessary to take into account the renormalization term $M_1(m,\mu)$.

We pass to euclidian metric and carry out the integral over angle.. Here p^2 and q^2 can be neglected as compared with k^2, and p^2 as compared with q^2 (but not **q** with respect to **p** in the term **p** + **q**). The limit of the q^2 integration can be taken as p^2 or k^2, which is not essential. We get the result

$$s_1(\xi,\eta)=\frac{1}{2}\lambda s_0^2(\eta)F(\eta)\int_{\xi}^{\eta}[s_0(z) \tag{30}$$
$$+2s_1(\xi,z)]F(z)\varphi(z)\,dz,$$

where

$$\xi=\ln\left(-\frac{p^2}{m^2}\right);\ \eta=\ln\left(-\frac{k^2}{m^2}\right).$$

Equation (30) can be solved to yield $s_1(\xi\ \eta)$ in terms of s_0, F and ϕ. To do this conveniently[1], we introduce the function

$$q(\xi,\eta)=\frac{s_1(\xi,\eta)}{s_0^2(\eta)F(\eta)}. \tag{31}$$

From Eqs. (30) and (25), we find easily

$$\frac{\partial}{\partial\eta}[s_0(\eta)q(\xi,\eta)]=\frac{1}{2}\lambda s_0^2(\eta)F(\eta)\varphi(\eta), \tag{32}$$

whence

$$s_0(\eta)q(\xi,\eta)=\frac{1}{2}\lambda\int_{\xi}^{\eta}s_0^2(z)F(z)\varphi(z)\,dz; \tag{33}$$

from which we learn that $q(\xi\ \xi)=0$. Thus,

$$s_1(\xi,\eta)=\frac{1}{2}\lambda s_0(\eta)F(\eta)\int_{\xi}^{\eta}s_0^2(z)F(z)\varphi(z)\,dz. \tag{34}$$

Now with the help of Eq. (34), we can prove the correctness of our previous assertion that the substitution of s_1 in the equation for the nucleon Green's function leads to an integral which diverges even after subtraction of the terms $M_0(m)$ and $(\mathbf{p}-m)M_0'(m)$. In fact, we differentiate Eq. (23) with respect to ξ. Then, if the subtraction of $M_0(m)$ and $(\mathbf{p}-m)M_0'(m)$ were to make the integral converge, after one (or at most, two) differentiations, we should get a finite expression. However, as is not difficult to see from Eqs. (23) and (34), after an arbitrary number of differentiations, the integral on ξ in Eq. (23) will give rise to logarithmic divergences. That confirms our assertion that momenta satisfying $|k^2|\gg|p^2|$ play a fundamental role in that part of the mass operator which contains $s_1(p^2,k^2)$.

3. EQUATIONS FOR THE MESON GREEN'S FUNCTION

We consider first of all the small correction to Γ_5, which must be taken into account in solving Eqs. (12) and (13) for the Green's function of the meson. Because the polarization operator diverges quadratically, small corrections to Γ_5, decreasing as $1/p^2$, still lead to logarithmically diverging integrals and must be taken into account. On the other hand, the function s_1 can be neglected in the polarization operator. In fact, in Eq. (13)

there appears $\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}, \mathbf{k})$ in which the additions will be small for the case which interests us, $|p^2| \gg |k^2|$. Thus, in complete analogy with the case of the mass operator, the substitution of the small corrections to Γ_5 into the polarization operator leads to an integral which diverges after the subtraction of $P_0(\mu^2)$ and $(k^2-\mu^2)P_0'(\mu^2)$ ["*b*-divergences" in the polarization operator which are compensated by the factor Z_1 in Eq. (13)]. Therefore, it is not necessary to take into account the function $s_1(p^2, k^2)$ which approaches zero for equal nucleon momenta.

Consequently, it is necessary to substitue $\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}, \mathbf{k})$ into Eq. (13) in the following form:

$$\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k}) = \gamma_5\left[s_0 + \frac{k^2}{p^2+k^2}\, s_2(k^2, p^2)\right], \tag{35}$$

where $s_2(k^2, p^2)$ is a slowly varying function of its arguments. It can be shown that it is not necessary to take into account small corrections of the form $[\mathbf{kp}(kp)/p^4]\, s_3$. In fact, considering the inhomogeneous term in the equation for s_3, i.e., the expression obtained by replacing Γ_5 by $\gamma_5 s_0$ on the right hand side of Eq. (9), one can convince oneself that in the factor of $\mathbf{kp}(kp)/p^4$ there does not occur $\ln(p^2/k^2)$ for $|p^2| \gg |k^2|$. This means that it is not necessary to take into account additional terms in Γ_5 of such a form.

We construct the equation for $s_2(k^2, p^2)$ for $|p^2| \gg |k^2|$. We write $M_1(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k})$ indicating all momenta explicitly.

$$\begin{aligned} M_1(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k}) = {}& \\ & -\frac{g^2}{4\pi^3 i}\int \Gamma_5(\mathbf{p}, \mathbf{p}-\mathbf{q}; -\mathbf{q})\, G(\mathbf{p}-\mathbf{q}) \\ & \times \Gamma_5(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}+\mathbf{k}; \mathbf{k})\, G(\mathbf{p}-\mathbf{q}+\mathbf{k}) \\ & \times \Gamma_5(\mathbf{p}-\mathbf{q}+\mathbf{k}, \mathbf{p}+\mathbf{k}; \mathbf{q})\, D(q^2)\, d^4q. \end{aligned} \tag{36}$$

The function $s_2(k^2, p^2)$ is equal to zero (to logarithmic approximation) if the momentum of the meson k^2 is of the same order of magnitude as the momentum of the nucleon p^2 (this is suggested by perturbation theory and confimed by the following results). Thus, it follows that in Eq. (36) the function s_2 need appear only in the middle one of the three Γ_5, and in the two outside ones, one can set $\Gamma_5 = \gamma_5 s_0$. We make the change of variable $\mathbf{p}-\mathbf{q} = \mathbf{q}'$ in Eq. (36). The rapidly varying factor in the functions G, D will equal

$$\begin{aligned} & (\mathbf{p}-\mathbf{q})^{-1}(\mathbf{p}-\mathbf{q}+\mathbf{k})^{-1}q^{-2} \\ & \qquad = \left[1 + \frac{\mathbf{q}'\boldsymbol{k}}{q'^2}\right]\frac{1}{(q'+k)^2(p+q')^2}. \end{aligned} \tag{37}$$

Bearing in mind that $|k^2| \ll |q'^2| \ll |p^2|$, it is easy to figure out the arguments of all slowly varying functions:

$$\begin{aligned} & M_1(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k}) \\ & \quad = \frac{g^2}{4\pi^3 i}\gamma_5 \int\left[1 + \frac{\mathbf{q}'\mathbf{k}}{q'^2}\right]\frac{d^4q'}{(q'+k)^2(p+q')^2} \\ & \quad \times s_0^2(p^2)\, F^2(q'^2)\, \varphi(p^2)\left[s_0(q'^2) + \frac{k^2}{q'^2}\, s_2(k^2, q'^2)\right]. \end{aligned} \tag{38}$$

In the inhomogeneous terms of the equation for s_2 [i.e., those containing $s_0(q'^2)$] only the second term in the square bracket of Eq. (37) can give a contribution proportional to $(k^2/p^2)\ln(p^2/k^2)$. On the other hand, in terms proportional to s_2 in Eq. (38), one need only keep the first term in the square brackets in Eq. (37). Thus, we get the following equation for the function s_2:

$$\begin{aligned} & \frac{k^2}{p^2}\, s_2(k^2, p^2) \\ & \quad = \frac{g^2}{4\pi^3 i}\int \frac{d^4q'}{(q'+k)^2(p+q')^2}\, s_0^2(p^2)\, F^2(q'^2)\, \varphi(p^2) \\ & \qquad \times\left[\frac{\mathbf{q}'\mathbf{k}}{q'^2}\, s_0(q'^2) + \frac{k^2}{q'^2}\, s_2(k^2, q'^2)\right], \end{aligned} \tag{39}$$

where the integration should be from $|q'| \sim |k|$ to $|q'| \sim |p|$. Making the transition to euclidian metric and integrating over angle, we get the following one-dimensional integral equation for s_2:

$$\begin{aligned} s_2(\xi, \eta) = -\frac{\lambda}{2}\, s_0^2(\eta)\, \varphi(\eta) \int_\xi^\eta [s_0(z) & \\ - 2s_2(\xi, z)]\, F^2(z)\, dz, & \end{aligned} \tag{40}$$

where $\xi = \ln(-k^2/m^2)$; $\eta = \ln(-p^2/m^2)$. The solution of Eq. (40) is carried out analogously to the solution of Eq. (30) for s_1. We set

$$p(\xi, \eta) = \frac{s_2(\xi, \eta)}{s_0^2(\eta)\, \varphi(\eta)}. \tag{41}$$

Then

$$\frac{\partial}{\partial\eta}[s_0(\eta)\, p(\xi, \eta)] = -\frac{\lambda}{2}\, s_0^2(\eta)\, F^2(\eta) \tag{42}$$

and

$$s_0(\eta)\,p(\xi,\eta) = -\frac{\lambda}{2}\int_{\xi}^{\eta} s_0^2(z)\,F^2(z)\,dz. \quad (43)$$

Thus

$$s_2(\xi,\eta) = -\frac{\lambda}{2}\,s_0(\eta)\,\varphi(\eta)\int_{\xi}^{\eta} s_0^2(z)\,F^2(z)\,dz. \quad (44)$$

From Eq. (44) it is seen that $s_2(k^2, p^2)$ approaches zero for $|k^2| \sim |p^2|$. In addition, it is easily verified, that after the substitution of s_2 in the polarization operator, an infinite expression is obtained even after the subtraction of the quantities $P_0(\mu^2)$ and $(k^2-\mu^2)P_0'(\mu^2)$.

Finally, we reduce the equation for the Green's function of the meson to one-dimensional form. Here there is a complete analogy with the structure of the one-dimensional equation for the Green's function of the nucleon.

If $\gamma_5 s_0$ is taken in place of $\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k}; \mathbf{k})$, then the resulting integral will converge for $|p^2| \sim |k^2|$ thanks to the renormalization [subtraction of the quantities $P_0(\mu^2)$ and $(k^2-\mu^2) \times P_0'(\mu^2)$]. Here the terms $P_0(k^2)$ and $P_0(\mu^2)$ can be neglected on the same grounds as in the equation for $G(\mathbf{p})$. The remaining term, $(k^2-\mu^2) \times P_0'(\mu^2)$ reduces to a one-dimensional integral by repeated use of our method and gives

$$-(k^2-\mu^2)\,P_0^{(0)'}(\mu^2) = 4\lambda Z_1 k^2\int_0^{\xi} s_0(z)\,F^2(z)\,dz. \quad (45)$$

In the integral, the quantity $\gamma_5(k^2/p^2)s_2(k^2,p^2)$ which replaces $\Gamma_5(\mathbf{p}, \mathbf{p}+\mathbf{k};\mathbf{k})$ plays a role for momenta $|p^2| \gg |k^2|$. Proceeding in precisely the same manner as in the derivation of the equation for $G(\mathbf{p})$, we get

$$P_0^{(1)}(k^2) - (k^2-\mu^2)\,P_0^{(1)'}(\mu^2) = -8\lambda Z_1 k^2\int_{\xi}^{\infty} F^2(z)\,s_2(\xi,z)\,dz + 8\lambda Z_1 k^2\int_0^{\infty} F^2(z)\,s_2(0,z)\,dz. \quad (46)$$

Gathering together Eqs. (12), (13), (45) and (46), we get the one-dimensional integral equation for the Green's function of the meson

$$1 + 4\lambda Z_1\int_{\xi}^{\infty} F^2(z)\,[s_0(z) - 2s_2(\xi,z)]\,dz - 4\lambda Z_1\int_0^{\infty} F^2(z)\,[s_0(z) - 2s_2(0,z)]\,dz = \frac{1}{\varphi(\xi)}. \quad (47)$$

4. THE ASYMPTOTIC GREEN'S FUNCTION OF NUCLEON AND MESON

We solve the system of Eqs. (23), (25), (30), (40) and (47). With the help of Eqs. (30) and (31), we write Eq. (23) thus:

$$\frac{1}{F(\xi)} = 1 + 3Z_1 q(\xi,\infty) - 3Z_1 q(0,\infty), \quad (48)$$

and, with the help of Eqs. (40) and (41), Eq. (47) thus:

$$\frac{1}{\varphi(\xi)} = 1 - 8Z_1 p(\xi,\infty) + 8Z_1 p(0,\infty) \quad (49)$$

Differentiating Eqs. (48) and (49) with respect to ξ and using Eqs. (33), (43) and (26), we find

$$\frac{d}{d\xi}\frac{1}{F(\xi)} = 3Z_1\frac{\partial q(\xi,\infty)}{\partial\xi} = -\frac{3}{2}\lambda s_0^2(\xi)\,F(\xi)\,\varphi(\xi), \quad (50)$$

$$\frac{d}{d\xi}\frac{1}{\varphi(\xi)} = -8Z_1\frac{\partial p(\xi,\infty)}{\partial\xi} = -4\lambda s_0^2(\xi)\,F^2(\xi), \quad (51)$$

while differentiation of Eq. (25) gives

$$\frac{ds_0(\xi)}{d\xi} = -\lambda s_0^3(\xi)\,F^2(\xi)\,\varphi(\xi). \quad (52)$$

The initial conditions for the differential Eqs. (50) - (52) follow from the integral Eqs. (23), (25) and (47):

$$F(0) = s_0(0) = \varphi(0) = 1. \quad (53)$$

Equations (50) - (52) are easily solved. The solutions corresponding to the initial conditions (53), have the form:

$$F(\xi) = (1-5\lambda\xi)^{-3/10}, \quad (54)$$
$$s_0(\xi) = (1-5\lambda\xi)^{1/5}$$
$$\varphi(\xi) = (1-5\lambda\xi)^{-4/5}:$$

The substitution of these expressions (54) in the integral equations shows thay they are indeed solutions, provided that the path of integration around the singularities is chosen consistent with

$$\xi = \ln\left(-\frac{p^2}{m^2(1-i\varepsilon)}\right) = \ln\left(-\frac{p^2}{m^2}\right) + i\varepsilon,$$

where $\epsilon > 0$.

For small $\lambda\xi$, one can expand Eq. (54) in series in $\lambda\xi$. The result of a calculation of G, Γ_5 and D by perturbation theory, coincides with the series for Eq. (54) (this was verifed up to terms of second order inclusively). We mention that in the solution of the integral equation the factor Z_1 cancels, and the solution (54) appears finite and free from ambiguities of any kind.

The solution (54) for the Green's function of nucleon and meson has a pole * at the point $\ln(-p^2/m^2) = 1/(5\lambda)$. Physically, the existence of such a pole indicates the appearance of a meaningless solution, because the mass κ, corresponding to such a pole is imaginary [in order that $p^2 = \kappa^2$ may satisfy the equation $\ln(-\kappa^2/m^2) = 1/(5\lambda)$]. The meaninglessness of the solution near or at the pole can be seen from another point of view. The experimental charge g is connected with the fictive charge g_0 by the correspondence $g_0^2 = Z_1^2 Z_2^{-2} Z_3^{-1} g^2$. Clearly, both the experimental charge g and the fictive charge g_0 should be real quantities (the conclusion follows from the hermitean character of the interaction hamiltonian). Consequently, the product $Z_1 Z_2^{-1} Z_3^{-1/2}$ is necessarily real and positive. But, calculating it with the aid of the solution (54), and Eqs. (5) and (55), one can convince oneself that $Z_1 Z_2^{-1} Z_3^{-1/2}$ is complex if Eq. (54) is taken sufficiently close to its pole.

Thus, we arrive at a contradiction whose resolution is that the solution (54) is valid only up to the pole. In fact, considering the cross-over diagrams we have neglected, for example, the one depicted in Fig. 3, one can see that the ratio of the contribution from them and the contribution from the diagrams we have taken into account will be of order of magnitude

$$\frac{\lambda}{(1-5\lambda\xi)^{\alpha}},$$

where α is some positive fractional number. Therefore, close to the pole where $1 - 5\lambda\xi \lesssim \lambda$, the contribution from the cross-over diagram will not be small, and our solutions lose their meaning. Physically, this corresponds to the fact that at small distances, interaction with a small coupling constant (which would give weak interaction at ordinary distances) becomes strong. Thus, for the construction of the asymptotic Green's functions for such large momenta it is necessary to take into account a significantly larger number of diagrams than we have done when we omitted all cross-over diagrams.

A more precise determination of the limits of applicability at high momenta of the formula we have obtained requires additional analysis, since, for $1 - 5\lambda\xi \sim \lambda$ in the expansion of Eq. (54) in powers $(\lambda\xi)^n$, terms with $n \sim \lambda^{-1}$ turn out to be essential. In such a situation the asymptotic series of perturbation theory already begin to diverge. The circumstance that the method of renormalization considered in reference 2 led to the removal of infinities in the above problem permits one to hope that it may be applicable in the solution of more complicated problems.

In conclusion, the authors express their gratitude to Academician L. D. Landau and to A. A. Abrikosov and I. M. Khalatnikov for the opportunity to see the manuscript of reference 1 before publication and also for useful discussions.

Translated by A. S. Wightman
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* In neutral pseudoscalar theory, the solution has the following form:

$$F(\xi) = (1 - 5\lambda\xi)^{-1/10};$$
$$s_0(\xi) = (1 - 5\lambda\xi)^{-1/5};$$
$$\varphi(\xi) = (1 - 5\lambda\xi)^{-2/5}.$$

A pole also appears in this variant of the theory. It seems that the presence of this pole is due to the same general cause as in symmetrical theory.

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The Effect of Interelectronic Collisions on the Electrical Conductivity and Skin Effect in Metals

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The effect of collisions between electrons on the electrical conductivity, thermal conductivity and skin effect in metals is considered. The question of electronic viscosity is discussed. It is shown that taking account of interelectronic collisions does not introduce notable changes into the results of the theory of the anomalous skin effect in metals.

IN a recent article[1] the assertion was made that the electronic viscosity in metals, considered as the result of interelectronic interaction, is essential in the calculation of the absorbing power of metals in the infrared region of the spectrum. If this assertion were true, then it would be of undoubted interest, especially in view of the problems currently confronting the optics of metals[2]. Hence we have attempted to consider in more detail the question of the effect of the interelectronic interaction on the optical properties of metals. This is especially necessary in connection with the series of unclear or doubtful moments present in reference 1. The question considered here is also closely connected with the question of the influence of the interelectronic interaction on the electrical conductivity and other kinetic coefficients. Thus in Part 1 of the present work, the effect of interelectronic collisions on electric and thermal conductivity is considered, in Part 2, the problem of electronic viscosity is considered and, finally Part 3 is devoted to the explanation of the effect of collisions between electrons on the skin effect in metals, and, in particular, on their absorbing power. In addition to this it is shown that the method and the results of reference 1 are incorrect and that for frequencies which are not too high it is not essential that the interelectronic interaction be taken into account.

1. In the theory of the electrical conductivity of metals the interaction between electrons is not ordinarily taken into account explicitly, in spite of the fact that the corresponding energy of interaction is of the same order as the energy of interaction of the electrons with the lattice. Until now no strict theoretical basis for the possibility of successfully using such an approximation (in the case of metals not in a superconducting state) has been given. However, there is no doubt that the effect of the interaction between electrons is greatly weakened in view of the Pauli principle. The success of the shell representation in the case of nuclear theory is analogously explained. Of course, the operation of the Pauli principle cannot reduce the effect of the interelectronic interaction to exactly zero, since in the zone where the Fermi distribution is washed out electrons may change their state. Together with this it is natural to think that in this region the interelectronic interaction might be taken account of by the method of the kinetic equation and considered as a perturbation. In what follows we shall do just this, analogously to the way in which it has been done in earlier works[3-5], to explain the effect of interelectronic collisions on the statistical electrical conductivity. In order to facilitate the exposition and also in order to be able to make a few observations, we shall first dwell further on this question.

Within the framework of elementary representations the length of the free path for interelectronic collisions can be written directly

$$l_{ee} \sim \frac{1}{q_{ee} n_0 (kT/E_0)^2}, \qquad \nu_{ee} = \frac{1}{\tau_{ee}} = \frac{v_0}{l_{ee}} \sim q_{ee} n_0 \left(\frac{kT}{E_0}\right)^2 v_0, \tag{1}$$

where n_0 is the concentration of conduction electrons, $E_0 = \frac{1}{2} m v_0^2$ is the energy at the Fermi boundary and q_{ee} is the corresponding effective cross section. (For electrons which are not free the role of n_0/E_0 is played by the density of states $2\,dN/dE$). The meaning of the expressions (1) becomes completely clear if we take account of the fact that an electron lying in the zone where the Fermi distribution washes out can collide only with one of the electrons in that same region, but the concentration of such electrons is $\sim n_0(kT/E_0)$. A second factor of kT/E_0 appears in Eq. (1) in virtue of the Pauli principle being taken into account for the final state (both electrons must also be in the washed out zone after the collision).

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[1] C. W. Benthem and R. Kronig, Physica **20**, 293(1954)

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[3] L. D. Landau and I. Ia. Pomeranchuk, J. Exper. Theoret. Phys. USSR **7**, 379 (1937)

[4] W. G. Baber, Proc. Roy. Soc. (London) **A158**, 383 (1937)

[5] S. V. Vonsovskii and A. A. Berdyshev, J. Exper. Theoret. Phys. USSR **25**, 723 (1953)

For purposes of estimation we may take as q_{ee} either the Coulomb cross section $q'_{ee} = (\pi e^4/E_r^2)\ln(E_r\rho/e^2)$ or the cross section $q''_{ee} \sim \pi n_0^{-2/3}$, where E_r is the energy of the relative motion and ρ is the radius of cutoff (of screening). Since $\rho \sim n_0^{-1/3} \sim 3 \times 10^{-8}$ cm and $E_r \sim E_0 \sim 10^{-11} - 10^{-12}$erg, then $q'_{ee} \sim \pi e^4/E_0^2 \lesssim 3 \times 10^{-15}\mathrm{cm}^2$ and $q''_{ee} \sim 3 \times 10^{-15}\mathrm{cm}^2$. The numerical calculations carried out by Abrahams[6] for $E_0 \approx 3.5$ ev give $q_{ee} \approx 1.5 \times 10^{-15}\mathrm{cm}^2$. Taking, for purposes of orientation, $q_{ee} \approx 10^{-15}\mathrm{cm}^2$, $n_0 \sim 3 \times 10^{22}\mathrm{cm}^{-3}$ and $E_0 \sim 3$ev (that is, $T_0 = E_0/k \sim 3 \times 10^4$ degrees), we obtain

$$l_{ee} \sim 3\cdot 10^{-8}(E_0/kT)^2 \sim 3\cdot 10^{-3}(10^2/T)^2 \ \mathrm{cm} \qquad (2)$$

For collisions of electrons with the lattice we find, if we take the path length $l_{el} \sim 3 \times 10^{-6}$ cm for $T \sim 300°$,

$$l_{el} \sim \frac{10^{-3}}{T}\ \mathrm{cm} \ \ (\text{for} \ \ T > \Theta \sim 10^2),$$

$$l_{el} \sim 10^{-6}\left(\frac{10^2}{T}\right)^5 \ \mathrm{cm} \ \ (\text{for} \ \ T < \Theta \sim 10^2). \qquad (3)$$

Values of l_{ee} and l_{el} are given in Table 1, along with the path length for collisions with impurities, which is taken, by way of example, as $l_{ei} \sim 3 \times 10^{-3}$cm (this corresponds to a very pure specimen).

From Table 1 it is clear that, beginning at 10 – 20° and below, interelectronic collisions may play a notable, or even a determining, role in the temperature dependent part of the resistance, which part is proportional to $(1/l_{el}) + (1/l_{ee})$. Experimental results[7] show that for Na at $T = 8 - 10°$ the contribution of interelectronic collisions to the temperature dependent part of the resistance does not exceed 10 – 20% (we here make use of Mattissen's law, which is probably valid in the given case). This result does not contradict the estimates cited, but indicates that with further precision in the experiment and in the corresponding calculations there may be hope of finding an explanation of the role of interelectronic collisions, for example, in Na. Such an explanation would have great significance. The fact is that the estimates cited may reasonably be applied to the calculation of electrical conductivity only in case the interelectronic collisions lead to a change in the current. By virtue of the law of conservation of momentum this does not occur for free electrons, and thus an expression of the type of Eq. (1) can be used in estimating the change in electrical conductivity only on the presupposition of the presence of exchange processes[3] or semiconductivity[4], when an electron collides with a "hole". Within the framework of existent representations the latter process cannot occur in univalent metals. Exchange processes for collisions of electrons with phonons also cannot occur* in this case, while exchange processes for interelectronic collisions can occur only very seldom, for definite favorable conditions (see below). Hence it seems appropriate to estimate the effect of interelectronic collisions on the electrical conductivity both with account being taken of exchange processes and without account being taken of them. This is done below by the method of the kinetic equation.

Writing the distribution function in the form $f = f_0 + \phi$, where f_0 is the equilibrium (Fermi) function, we obtain in the linear approximation

TABLE I

T, degree	l_{ee}, cm	l_{el}, cm	l_{en}, cm
10^3	$3\cdot10^{-5}$	10^{-6}	$3\cdot10^{-3}$
300	$3\cdot10^{-4}$	$3\cdot10^{-6}$	$3\cdot10^{-3}$
100	$3\cdot10^{-3}$	—	$3\cdot10^{-3}$
30	$3\cdot10^{-2}$	$3\cdot10^{-4}$	$3\cdot10^{-3}$
10	0.3	0.1	$3\cdot10^{-3}$
1	30	10^4	$3\cdot10^{-3}$

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[6] E. Abrahams, Phys. Rev. 95, 839 (1954)

[7] D, K. C. MacDonald and K. Mendelssohn, Proc. Roy. Soc. (London) A202, 103 (1950)

*The question as to whether these processes actually occur is not completely clear, but according to Klemens[8] (see also Wilson[9]) there are bases for supposing that a stable state of the lattice may perhaps be established even without exchange processes.

[8] P. G. Klemens, Proc. Phys. Soc. (London) A64, 1030 (1951)

[9] A. H. Wilson, *The Theory of Metals*, (Cambridge, 1953)

$$i\omega\varphi + \frac{\mathbf{p}}{m}\frac{\partial\varphi}{\partial\mathbf{r}} - e\vec{\mathcal{E}}\frac{\partial f_0}{\partial\mathbf{p}} = -J_{el} - J_{ee}, \tag{4}$$

where the field $\vec{\mathcal{E}}(t) = \vec{\mathcal{E}}e^{i\omega t}$, m is the effective mass of the electron, J_{el} is the integral of the collisions of the electrons with the lattice (with phonons), which is presumed to be in equilibrium at a certain temperature T, and J_{ee} is the integral of the collisions of the electrons with each other.

Concerning ourselves with the electrical conductivity, we seek a solution of Eq. (4) in the form:

$$\varphi = e\vec{\mathcal{E}}\mathbf{p}A(p). \tag{5}$$

Relation (5) leads to the following expression for the total current density

$$j_t = j + i\omega P$$

$$= \sigma'(\omega)\mathcal{E} = \left(\sigma + \frac{\varepsilon - 1}{4\pi}i\omega\right)\mathcal{E} = \frac{i\omega}{4\pi}(\varepsilon' - 1)\mathcal{E},$$

where j is the conduction current density, P the polarization, σ' the complex electrical conductivity, ϵ the dielectric constant, σ the electrical conductivity and $\epsilon' = \epsilon - i(4\pi\sigma/\omega)$ the complex dielectric constant. The magnitudes of ϵ, σ and ϵ' depend on ω; we may also designate by σ_0 the magnitude of the statistical electrical conductivity $\sigma'(0) = \sigma(0)$.

Considering a field not very different from homogeneous, we neglect the space derivatives of the electric field which arise when Eq. (5) is substituted into Eq. (4). The following equation, determining $A(p)$, is then obtained from Eq. (4):

$$\partial f_0/\partial\mathbf{p} = J_{ee}[\mathbf{p}A(p)] + [\nu_{el}(p) + i\omega]\mathbf{p}A(p). \tag{6}$$

As a result of Eq. (6) it is considered that in the high and low temperature regions which are of interest to us:

$$J_{el}[\mathbf{p}A(p)] = \nu_{el}(p)\mathbf{p}A(p), \tag{7}$$

where the number of collisions with the lattice (with phonons and impurities) $\nu_{el}(p)$ is a certain function of p.

Equation (6) is somewhat complicated. However, the situation becomes essentially simplified in the case where the interaction of the electrons with each other may be considered weak, that is, where it changes the magnitudes of σ and ϵ only slightly. Then, applying the method of successive approximations, one can obtain the following expression for $A(p)$:

$$A(p) = \frac{\mathbf{p}\partial f_0/\partial\mathbf{p}}{p^2[\nu_{el}(p) + i\omega]}$$

$$- \frac{p_l}{p^2[\nu_{el}(p) + i\omega]}J_{ee}\left(\frac{p_l p_j \partial f_0/\partial p_j}{p^2[\nu_{el}(p) + i\omega]}\right). \tag{8}$$

Neglecting collisions of electrons with each other, that is, neglecting J_{ee}, and assuming that f_0 depends only on the energy E of the electron, we find that Eq. (8) leads to the well-known formula for σ'

$$\sigma' = \sigma'_{el} = \frac{2e^2}{3}\frac{v_0^2}{\nu_{el}(p_0) + i\omega}\left(\frac{dN}{dE}\right)_{E_0}$$

$$= \frac{e^2 n_0}{m[\nu_{el}(p_0) + i\omega]}, \tag{9}$$

where $2\,dN/dE$ is the density of states and n_0 is the concentration of conduction electrons.

The change in σ' arising from the interaction of the electrons with each other is determined by an integral of the form:

$$I = \frac{2}{(2\pi h)^3}\int d\mathbf{p}_1\,\Phi(\mathbf{p}_1)\,J_{ee}\left[\psi(\mathbf{p}_1)\frac{\partial f_0}{\partial E_1}\right], \tag{10}$$

where ψ and Φ are certain functions of the electron's momentum $\mathbf{p}_1$[the form of the function ψ is clear from Eq. (8)], E_1 is the energy of the electron and the integral of the collisions of electrons with electrons has the form:

$$J_{ee}\left[\psi(\mathbf{p}_1)\frac{\partial f_0}{\partial E_1}\right] \tag{11}$$

$$= \frac{1}{kT}\sum_n\int d\mathbf{p}_2\,d\Omega_{\mathbf{p}'_1}\,q_{ee}(|\mathbf{p}_1 - \mathbf{p}_2|, \vartheta)\left|\frac{\mathbf{p}_1 - \mathbf{p}_2}{m}\right|$$

$$\times f_0(p_1'^2)\,f_0(p_2'^2)\,[1 - f_0(p_1^2)]\,[1 - f_0(p_2^2)]$$

$$\times [\psi(\mathbf{p}_1) + \psi(\mathbf{p}_2) - \psi(\mathbf{p}'_1) - \psi(\mathbf{p}'_2)],$$

where $q_{ee}(p, \vartheta)$ is the cross section for collisions of electrons with electrons, for which ϑ is the angle characterizing the scattering. From the laws of conservation of energy and quasi-momentum of the electrons**

$$\mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - 2\pi\mathbf{g}\,n. \tag{12}$$

$$p_1'^2 = p_1^2 + p_2^2 - (\mathbf{p}_1 + \mathbf{p}_2 - 2\pi\mathbf{g}\,n)^2,$$

** It is assumed that during the scattering the electrons remain in the same zone. The energy is regarded as a quadratic function of the momentum ($E = p^2/2m_{\text{eff}}$).

where $\mathbf{g}/\hbar$ is the vector of the reciprocal lattice and n is an integer.

We shall consider integral (10) without specifying the form of the functions Φ and ψ for the present, since we have also in view using the resulting formulas in a different way than in their application to the calculation of the electrical conductivity. Taking into account Eqs. (11) and (12), we write Eq. (10) in the following form

$$
\begin{aligned}
I = {} & \frac{4}{(2\pi\hbar)^6\, mkT}\int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{s} \\
& \times\; q_{ee}(|\mathbf{p}_1-\mathbf{p}_2|, \cos\vartheta)\,|\mathbf{p}_1-\mathbf{p}_2|\,[1-f_0(p_1^2)] \\
& \times [1-f_0(p_2^2)]\, f_0\left\{\frac{p_1^2+p_2^2}{2}\right. \qquad (13) \\
& \qquad + \frac{1}{2}\sqrt{2(p_1^2+p_2^2)-(\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n)^2} \\
& \times |\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n|\cos\vartheta\left.\vphantom{\frac{1}{2}}\right\} \times f_0\left\{\frac{p_1^2+p_2^2}{2}\right. \\
& \qquad - \frac{1}{2}\sqrt{2(p_1^2+p_2^2)-(\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n)^2} \\
& \times |\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n|\cos\vartheta\left.\vphantom{\frac{1}{2}}\right\}\Phi(\mathbf{p}_1) \\
& \qquad \times \left\{\psi(\mathbf{p}_1)+\psi(\mathbf{p}_2)-\psi\left(\frac{\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n}{2}\right.\right. \\
& + \frac{1}{2}\sqrt{2(p_1^2+p_2^2)-(\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n)^2}\,\mathbf{s}\left.\vphantom{\frac{1}{2}}\right) \\
& - \psi\left(\frac{\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n}{2}\right. \\
& \qquad \left.\left. - \frac{1}{2}\sqrt{2(p_1^2+p_2^2)-(\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n)^2}\,\mathbf{s}\right)\right\}.
\end{aligned}
$$

Here $\mathbf{s}$ is a unit vector, ϑ is the angle between $\mathbf{s}$ and $\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n$. In the integration with respect to $\cos\vartheta$ the principal contribution to the integral is made by the region $\cos\vartheta \sim 0$; hence in the slowly varying functions we may set $\cos\vartheta = 0$ and extend the integral with respect to $y = \cos\vartheta$ from $-\infty$ to $+\infty$; in the same way we can set $p_1 = p_2 = p_0$ in the slowly varying functions. We thus obtain

$$
\begin{aligned}
I = {} & \frac{32\,\zeta(3)}{(2\pi)}\,\frac{m^2 p_0^2 (kT)^2}{\hbar^6}\int d\Omega_{\mathbf{p}_{10}}\, d\Omega_{\mathbf{p}_{20}} \int_0^{2\pi} d\varphi_{\mathbf{s}_0} \times \\
& \times \frac{q_{ee}(|\mathbf{p}_{10}-\mathbf{p}_{20}|, 0)\,|\mathbf{p}_{10}-\mathbf{p}_{20}|}{|\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n|\sqrt{4p_0^2-(\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n)^2}}
\end{aligned}
$$

$$
\begin{aligned}
& \times\; \Phi(\mathbf{p}_{10})\left[\psi(\mathbf{p}_{10})+\psi(\mathbf{p}_{20})-\psi\left(\frac{\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n}{2}\right.\right. \\
& + \frac{1}{2}\sqrt{4p_0^2-(\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n)^2}\,\mathbf{s}_0\left.\vphantom{\frac{1}{2}}\right) \qquad (14) \\
& - \psi\left(\frac{\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n}{2}\right. \\
& \qquad \left.\left. - \frac{1}{2}\sqrt{4p_0^2-(\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n)^2}\,\mathbf{s}_0\right)\right] \\
& + O\left[\left(\frac{kT}{E_0}\right)^3\right],
\end{aligned}
$$

where $\mathbf{s}_0 = \mathbf{s}(\cos\vartheta = 0)$ and $\zeta(3) = 1.202$ [$\zeta(z)$ is the Riemann function].

If exchange processes are neglected or if they are impossible, when $n = 0$ in Eq. (12), the principal member of I, proportional to T^2, is equal to zero. As a result, the change in electrical conductivity appears to be reduced by a factor $\sim kT/E_0$. In the interesting temperature region $T \sim 10°$ this factor is of the order of 3×10^{-4} (for $T_0 = E/k \sim 3 \times 10^{-4}$), and the effect of interelectronic collisions on σ' can be considered practically equal to zero[‡]. Exchange processes for collisions of the electrons are possible according to Eq. (12) only if $4p_0 \geq 2\pi gn$, since all the momenta p_1, p_2, p_1' and p_2' are near p_0. In the one electron model for a univalent metal, half the conduction zone is filled and $4p_0 = 2\pi g$, that is, for $n = 1$ exchange processes are still possible but are already very difficult. Assuming that exchange processes occur only for $n = 1$, we obtain,

$$
\frac{\sigma' - \sigma'_{el}}{\sigma_{el}} \equiv \frac{\Delta\sigma'_{ee}}{\sigma'_{el}} = -\frac{\nu_{ee}}{\nu_{el} + i\omega}, \qquad (15)
$$

$$
\nu_{ee} = q_{\text{eff}}\, n_0 v_0 \left(\frac{kT}{E_0}\right)^2 = \frac{v_0}{l_{ee}},
$$

where $\Delta\sigma'_{ee}$ is the change in σ' connected with interelectronic collisions, $\nu_{el} = v_0/l_l$ is the number of collisions with the lattice $\left(\frac{1}{l_l} = \frac{1}{l_{el}} + \frac{1}{l_{ei}}\right)$ and

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[‡] We note[10] that for a non-degenerate electron-ion plasma, taking account of the interelectronic collisions changes the statistical electrical conductivity by an amount of the order of unity (σ_0 increases by a factor of 1.73). At high frequencies, when $\omega^2 \gg \nu^2$, the effect of interelectronic collisions is small even in the case of a plasma[11].

[10] R. Landshoff, Phys. Rev. 76, 904 (1949); 82, 442 (1951)

[11] V. L. Ginzburg, J. Tech. Phys. USSR 21, 943 (1951)

$$q_{\mathrm{eff}} = \frac{3\pi^2\zeta(3)}{x}\int\limits_{x-1}^{1} dz\, q_{ee}\left(2p_0\sqrt{1-(x-z)^2}, 0\right)$$

$$\times \sqrt{1-(x-z)^2}$$

$$\times \left\{\frac{\pi}{2} - \arcsin z + z\sqrt{1-z^2}\right\},$$

where $2 \geq x = \pi g/p_0 \gtrsim 1$.

To the same accuracy we may also write (assuming that $\nu_{ee}/\,|\nu_{el} + i\omega| \ll 1$)

$$\sigma' = e^2 n_0 / (m\,[\nu_{el} + \nu_{ee} + i\omega]). \qquad (16)$$

If exactly half the zone is filled, then $x = 2$, $q_{\mathrm{eff}} = 0$ and exchange processes may be said to occur only in the next approximation in terms of kT/E_0. This result is connected also with the assumption of the sphericity of the Fermi surface and the quadraticity of the dependence of the energy on the momentum. The absence of a contribution to the resistance of Na from electron-electron collisions would seem to show that the usual isotropic one electron model applies well to this case.

In order to estimate the maximum possible contribution from the exchange processes, we set $x \approx 1$. Then for the hard sphere model $q_{ee}(\theta) = \rho^2/4$ (ρ is the radius of a sphere) and $q_{\mathrm{eff}} = 0.97 \times 3\pi^2\zeta(3)\rho^2/4 = 2.7\pi\rho^2$; for the screened Coulomb field $[V_{12} = (e^2/r_{12})e^{-r_{12}/\rho}]$

$$q_{\mathrm{eff}} = 3\pi^2\zeta(3)\frac{e^4}{16E_0^2}\, y\left(\frac{\hbar}{p_0\rho}\right),$$

where the magnitudes of y in their dependence on the parameter $\hbar/p_0\rho$ are arranged in Table 2.

TABLE II

$\hbar/p_0\rho$	y
0.2	1.64
0.5	0.78
1.0	0.36
2.5	0.10
5.0	0.03

Setting $\rho = 2 \times 10^{-8}$ cm and $E_0 = 3$ev ($p_0 = 10^{-19}$), we obtain $q_{\mathrm{eff}} \approx 3 \times 10^{-15}$ in the case of the spheres, and $q_{\mathrm{eff}} \approx 4 \times 10^{-15}$ in the Coulomb case. In the estimates given earlier we set $q_{\mathrm{eff}} = q_{ee} \sim 10^{-15}$ (see reference 2), and consequently the magnitudes of l_{ee} given in Table 1 are acceptable and may even be increased several fold. In the latter case, the effect of interelectronic collisions would already be notable in Na. Thus there already exist known bases for supposing that exchange processes do not play a major role in the case of Na. Later experiments may in principle lead to a solution of the question concerning the role of exchange processes in Na and other metals as a result of the determination of the temperature dependence of σ_0 for low temperatures. However, in virtue of the impossibility of relying on Mattissen's law, it is evident that success is possible here only for samples with negligibly small impurities, when $l_{ei} \lesssim 0.1$ cm (such samples have not yet been obtained).

The effect of interelectronic collisions on the thermal conductivity of metals is incomparably less than in the case of the electrical conductivity. This is clear from the fact that the path length for collisions of electrons with phonons, which enters into the expression for the coefficient of thermal conductivity, is less than the corresponding path length in the case of electrical conductivity by a factor $\sim (\Theta/T)^2$, where Θ is the Debye temperature. At the same time, the contribution of interelectronic collisions to electrical and thermal conductivity is determined by path lengths which are of the same order of magnitude. As a result, interelectronic collisions may be considered to occur only for $T \sim 0.1°$, but they are non-essential even in this region, apart from effects connected with the heat conductivity of the lattice, the finite dimensions of the samples or of the crystalline particles composing them, and the like.

In spite of the small role which, as is clear from the above, is played by interelectronic collisions, nevertheless, from the point of view of their effect on exchange processes, these collisions may be extremely important. Thus, in a consideration of nonlinear processes in metals, interelectronic collisions cannot be neglected, generally speaking, and, for example, at high temperatures they lead to the possibility of using the concept of the kinetic temperature of the electrons in a metal even for large currents[12].

2. During a change in the density of the electron current from point to point, a viscosity effect, similar to that which takes place during the flow of ordinary gases, may appear. The question of electronic viscosity is discussed in references 13,

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[12] V. L. Ginzburg and V. P. Shabanskii, Dokl. Akad. Nauk SSSR 100, 445 (1955)

[13] A. S. Kompaneets, J. Exper. Theoret. Phys. USSR 9, 920 (1939)

14 and 1, wherein the point of departure is essentially the equation of motion of the electrons:

$$mn_0 \frac{d\mathbf{u}}{dt} + mn_0 \nu \mathbf{u} - \eta_{ee} \Delta \mathbf{u} = en_0 \vec{\mathcal{E}}, \tag{17}$$

where **u** is the average velocity of the electrons, η_{ee} is the coefficient of viscosity and $\nu = \nu_{el}$ is the effective number of collisions.

In the linear approximation and with the viscosity neglected, $\mathbf{j}_t = en_0 \mathbf{u} = \frac{e^2 n_0 \vec{\mathcal{E}}}{m(\nu_{el} + i\omega)} = \sigma' \vec{\mathcal{E}}$. Hence, considering the role of the viscosity to be small, we obtain from Eq. (17)

$$\mathbf{j}_t = \sigma' \vec{\mathcal{E}} + \xi_{ee} \Delta \vec{\mathcal{E}},$$
$$\xi_{ee} = \frac{\eta_{ee} \sigma'}{mn_0 (i\omega + \nu_{el})} = \frac{e^2 \eta_{ee}}{m^2 (i\omega + \nu_{el})^2}. \tag{18}$$

We shall evaluate the coefficient of viscosity η_{ee}. Let the velocity **u** be directed along the x axis and be changing along the z axis. Then for $\omega = 0$, as follows from elementary kinetic considerations, the density of the current of momentum transferred to the lattice is of the order of $\frac{mn_0 v_0}{3} l_l \frac{du}{dz} = \eta_{el} \frac{du}{dz}$, where $\frac{1}{l_l} = \frac{1}{l_{el}} + \frac{1}{l_{ei}}$ and the fact that $l_{ee} \gg l_l$ has been taken into account. We are now interested in the current of momentum transferred to the electrons, equal to $\eta_{ee} du/dz$. In order to find η_{ee}, it is evidently necessary to multiply the coefficient η_{el} (see above) by the number of collisions of the electrons with the lattice along the path l_l, that is, by l_l / l_{ee}. Thus (see also reference 1)

$$\eta_{ee} \sim \frac{mn_0 v_0}{3} l_l \frac{l_l}{l_{ee}}$$
$$\sim \frac{mv_0}{3q_{ee}} \left(\frac{E_0}{kT}\right)^2 \left(\frac{l_l}{l_{ee}}\right)^2 \equiv \frac{mv_0}{3q_{ee}} \beta \equiv \eta_0 \beta. \tag{19}$$

Similarly, for arbitrary frequency

$$\eta_{ee} \sim \frac{mn_0 v_0^2}{3} \frac{\nu_{ee}}{\omega^2 + \nu_{el}^2}, \quad \nu_{ee} = \frac{v_0}{l_{ee}}, \quad \nu_{el} = \frac{v_0}{l_l}. \tag{20}$$

The magnitude of $\eta_0 = mv_0 / 3q_{ee} \sim 10^{-5}$ and may be written in the form[14] $\eta_0 \sim (3/8) n_0 \hbar$, since $q_{ee} \sim n_0^{-2/3}$ and $n_0 \sim (mv_0/2\pi\hbar)^3$. Using the magnitudes of l_{ee} and l_l introduced in Part 1, we see that for high temperatures the coefficient $\beta \sim 1$ (for $T_0 = E_0/k \sim 3 \times 10^4$). The coefficient β increases as the temperature is lowered and reaches the magnitude $\beta \sim 10^3$ for $T \sim 20°$, but beyond this point it begins to fall and tends to zero for $T \to 0$. It follows from this that the use of references 14 and 1 of the constant magnitude $\eta_{ee} \sim \eta_0$ (that is, $\beta \sim 1$) is unsuitable in the region of low temperatures[+].

We shall not consider in further detail the question of the form of the function $N_{ee}(T)$, since the use of equations (17) – (18), where η_{ee} is the coefficient of viscosity connected with the interaction between electrons, is inadmissible. The fact is that even in the complete absence of interelectronic collisions, when $\eta_{ee} = 0$, in a slightly inhomogeneous field

$$\mathbf{j}_t = \sigma' \vec{\mathcal{E}} + \xi_{el} \Delta \vec{\mathcal{E}}, \tag{21}$$

where in the quasistatic case $\xi_{el} \sim \sigma l_l^2$. The appearance of the second term in Eq. (21) is explained by the fact that the current at a given point depends on the field in the region of this point within a radius of the order of the free path length, and the differential form of Ohm's law is correct only if $l_l^2 / \delta^2 \to 0$, where δ is the skin depth (the distance in which the field $\mathcal{E}$ changes notably). In other words, the term of type $\xi \Delta \vec{\mathcal{E}}$ appears as soon as account is taken of the anomalous character of the skin effect[16,17], without consideration of interelectronic collisions. If account be taken of these collisions, then it is only necessary to change the coefficient slightly, the corresponding correction being small in virtue of the fact that $l_{ee} \gg l_l$. Actually, when account is taken of the anomalous character of the skin effect and of interelectronic collisions

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[14] R. Kronig and J. Korringa, Physica 10, 406, 800 (1943); R. Kronig, Physica 15, 667 (1949)

[+] In reference 14 the magnitude of η_{ee} is determined not from kinetic considerations but from an equation obtained by comparing the electron viscosity formula for ionization losses of energy at high frequencies in metals with an analogous formula found by another method[15]. Without pausing to go into this question in detail, we note that the use in reference 14 of an equation of the type of Eq. (17) is completely invalid (even apart from the necessity of taking account of collisions with the lattice, which is spoken below). This is already clear from the fact that in reference 14 the equation of type (17) is applied under conditions where the velocity is changing within distances of 10^{-8} - 10^{-10} cm, which are significantly smaller not only than l_{ee} but also than l_l, while, as is known from kinetic theory, Eq. (17) may be used only if the velocity **u** changes but slightly in distances of the order of the free path length.

[15] H. A. Kramers, Physica 13, 401 (1947)

[16] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) A195, 336 (1948)

[17] R. B. Dingle, Physica 19, 311 (1953)

$$j_t = \sigma' \vec{\mathcal{E}} + \xi_{el} \Delta \vec{\mathcal{E}} + \xi_{ee} \Delta \vec{\mathcal{E}}, \tag{22}$$

where in the quasistatic case $\xi_{el} \sim \sigma' l_l^2$ and $|\xi_{ee}| \sim \sigma' l_l^2 \frac{l_l}{l_{ee}} \ll \xi_l$ [see Eqs. (18) and (19)].

Everything that has been said is confirmed by a more rigorous kinetic consideration, on which it is also pertinent to dwell briefly in order to complete the picture.

Assuming that the electric field is slightly inhomogeneous, and assuming also that the spatial inhomogeneity in the distribution of the electrons is not large, we shall, following the method worked out in reference 18, seek a solution of Eq. (4) in the form:

$$\varphi = e\vec{\mathcal{E}}\,\mathbf{p}\,A(p) + e\frac{\partial \mathcal{E}_k}{\partial r_j}\left(\frac{p_k p_j}{p^2} - \frac{\delta_{jk}}{3}\right)B(p) \tag{23}$$

$$+ e\frac{\partial^2 \mathcal{E}_k}{\partial r_l\,\partial r_j}\Big\{C(p)\,p_k\,\delta_{lj} + D(p)\,(p_l\,\delta_{kj} + p_j\,\delta_{kl})$$

$$+F(p)\Big[p_k\Big(p_l p_j - \frac{3}{5}p^2\delta_{lj}\Big) + p_l\Big(p_j p_k - \frac{3}{5}p^2\delta_{jk}\Big)$$

$$+ p_j\Big(p_k p_l - \frac{3}{5}p^2\delta_{kl}\Big)\Big]\Big\}.$$

We shall neglect higher derivatives of the electric field. Relation (23) leads to the following expression for the total current*

$$\mathbf{j}_t = \sigma' \vec{\mathcal{E}} + \xi \Delta \vec{\mathcal{E}}, \tag{22a}$$

where σ' is determined as a function of $A(p)$ (see reference 5), and ξ is determined by the formula

$$\xi = -\frac{e^2}{3m}\int p^2\,d\,\mathbf{p}\,C(p). \tag{24}$$

Putting Eq. (23) into Eq. (4) leads to Eq. (6) for $A(p)$ and to the following equations for $B(p)$ and $C(p)$:

$$\frac{p^2}{m}\left(\frac{p_j p_k}{p^2} - \frac{\delta_{jk}}{3}\right)A(p) = J_{ee}\left[\left(\frac{\delta_{jk}}{3} - \frac{p_j p_k}{p^2}\right)B(p)\right]$$
$$-\left(\frac{\delta_{jk}}{3} - \frac{p_j p_k}{p^2}\right)[\nu'_{el}(p) + i\omega]\,B(p), \tag{25}$$

$$\frac{\mathbf{p}}{5m}B(p) \tag{26}$$
$$= -J_{ee}[\mathbf{p}\,C(p)] - [\nu_{el}(p) + i\omega]\,\mathbf{p}\,C(p),$$

where $\nu_{el}(p)$ is determined by Eq. (17), while $\nu'_{el}(p)$ is determined in the following manner:

$$J_{el}\left[\left(\frac{\delta_{lj}}{3} - \frac{p_l p_j}{p^2}\right)B(p)\right] \tag{27}$$
$$= \nu'_{el}(p)\left(\frac{\delta_{lj}}{3} - \frac{p_l p_j}{p^2}\right)B(p).$$

Assuming, as was done during the solution of Eq. (6), that the effect of the interaction of the electrons on each other may be considered as a perturbation, we find the following expression for $C(p)$:

$$C(p) = \frac{p^2 A(p)}{5m^2[\nu_{el}(p) + i\omega][\nu'_{el}(p) + i\omega]} \tag{28}$$
$$+ \frac{p_k}{5p^2 m^2[\nu_{el}(p) + i\omega]}J_{ee}\left(\frac{p_k p^2 A(p)}{[\nu_{el}(p) + i\omega][\nu'_{el}(p) + i\omega]}\right)$$
$$- \frac{3p_l p_k J_{ee}\left[\left(\frac{\delta_{lk}}{3} - \frac{p_l p_k}{p^2}\right)\mathbf{p}\frac{\partial f_0}{\partial \mathbf{p}}(\nu_{el} + i\omega)^{-1}(\nu'_{el} + i\omega)^{-1}\right]}{10m^2p^2[\nu_{el}(p) + i\omega][\nu'_{el}(p) + i\omega]}.$$

It follows from this that if the interaction of the electrons on each other be neglected:

$$\xi_{el} = \frac{2e^2}{15}\frac{v_0^4(dN/dE)_{E_0}}{[\nu_{el}(p_0) + i\omega]^2[\nu'_{el}(p_0) + i\omega]} \tag{29}$$
$$= \frac{e^2 n_0^2 p_0^2}{5m^3[\nu_{el}(p_0) + i\omega]^2[\nu'_{el}(p_0) + i\omega]};$$

$$\eta = \eta_{el} = \xi_{el}[i\omega + \nu_{el}(p)]^2(m/e)^2 \tag{30}$$
$$= \frac{n_0 p^2}{5m[\nu'_{el}(p) + i\omega]}.$$

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* Here and above it is assumed for simplicity that $\operatorname{div}\vec{\mathcal{E}} = 0$, since this occurs even in the most interesting cases (skin effect for a flat boundary surface and the like). We note that, as is clear from Eqs. (22), (22a) and everything that has been said, Eq. (17) has a well-known meaning if $\eta = \eta_{el} + \eta_{ee}$ is put in place of η_{ee}. Then we may, of course, speak of "electronic viscosity", having in mind that it is determined by collisions of the electrons with the lattice.

[18] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, (Leipzig-Berlin, 1912)

The order of magnitude of ν'_{el} and ν_{el} is the same. Hence Eq. (30) for $\omega = 0$ gives

$$\eta_{el} \sim \frac{mn_0 v_0}{5} l_l ,$$ which coincides with the estimate made earlier. This same is related also, of course, to the magnitude $\xi_{dl} \sim \sigma_0 l_l^2 / 5$ (it being assumed that $\omega = 0$).

We shall now take account of interelectronic collisions, assuming that $\xi = \xi_{el} + \xi_{ee}$, where $|\xi_{ee}| \ll |\xi_{el}|$. The calculations here are analogous to those which led to Eq. (14). In them we shall not take account of exchange processes, since in the case under consideration the chief term occurring in an analysis in terms of powers of kT/E_0 is conserved even when exchange processes are neglected, so that these processes do not change the order of magnitude of ξ_{ee}. As a result,

$$\frac{\xi_{ee}}{\xi_{el}} = \frac{\eta_{ee}}{\eta_{el}} = -\frac{\nu'_{ee}}{\nu'_{el} + i\omega}, \tag{31}$$

where

$$\nu'_{ee} = q'_{\text{eff}}\, n_0 v_0 (kT / E_0)^2,$$

$$q'_{\text{eff}} = \frac{9\pi\zeta(3)}{2} \int_{-1}^{+1} dx \frac{q_{ee}(\sqrt{2}\, p_0 \sqrt{1-x}, 0)}{\sqrt{2}\sqrt{1+x}} \times \left(\frac{1}{6} - \frac{2}{3}x + \frac{1}{2}x^2\right).$$

Equation (31) is of the same order as Eq. (15) and, for example, in the statistical case $|\xi_{ee}/\xi_{el}| \sim l_l/l_{ee}$, in agreement with the result of an elementary consideration. Along with this it is not at once evident in the elementary calculation that the magnitude of ξ_{ee} is negative [we integrated this moment above; the fact that $\xi_{ee} < 0$ is clear from a comparison of Eq. (16) with (15), while taking account of the necessity of substituting $\nu'_{el} + \nu'_{ee}$ for ν'_{el} in Eq. (29)] . Thus, taking account of interelectronic interaction leads only to the appearance of a small correction to the coefficients σ' and ξ in Eq. (22a), and it is clear from Part 1 that this correction may be completely disregarded. Thus the effect of the "electronic viscocity" connected with interelectronic collisions is negligibly small, since it is completely masked by the corrections connected with the anomalous character of the skin effect.

3. Relations (22), (22a) and analogous relations are applicable only when the second term is small compared to the first, that is, under the condition $\sigma' \gg \xi/\delta^2$, where δ is a characteristic distance within which the field $\vec{\mathcal{E}}$ changes (usually δ is the skin depth). However, even under this condition Eq. (22) cannot be used to find the correction to the formula obtained in the theory of the normal skin effect. This is due to the fact that relations (22), (22a) are valid only far from the boundaries. When the boundaries are taken into account, there appear, generally speaking, terms of the order l_l / δ (see reference 17) in the expressions for j_t and the surface impedance. It is already clear from this that relations (22), (22a) cannot without further analysis be applied in the theory of the skin effect, no matter what boundary conditions be introduced. Moreover, the boundary condition proposed in reference 1 is manifestly incorrect. Thus it is proposed in reference 1 that at the boundary of a metal $j_t = 0$, whereas from the boundary conditions used in a more general kinetic approach to the problem a completely different result is obtained; to wit, if the reflection of the electrons from the boundary is specular, then, as may readily be shown, $j_t = \sigma'\mathcal{E}$ on the boundary for $l_l \to 0$, but for diffuse reflection $j_t = \frac{1}{2}\sigma'\mathcal{E}$ on the boundary. In the latter case, which more nearly corresponds to reality, the function $j_t(z)$ for small l_l increases rapidly with distance from the boundary (distance from the boundary is z , with current j_t directed in a perpendicular direction); at a certain distance $z \sim l_l$ $dj_t/dz = 0$, $j_t \approx \sigma'\mathcal{E}$, and subsequently j_t begins to fall**.

On the basis of what has been said it is clear that in the theory of the skin effect there are no bases for using Eq. (22), no matter what boundary conditions it fulfills, but that it is expedient to turn directly to the solution of the kinetic equation (see references 16, 17). Here, in particular in the interesting limiting case of the clearly expressed anomalous skin effect, when $l_l \gg \delta$, the results do not in general depend on the collisions of the electrons with the lattice and with each other. In the general case the role of interelectronic collisions may be taken into account by changing the number of collisions ν_{el} to $\nu_{el} + \nu_{ee}$. The corresponding correction is always small, since $\nu_{ee} \ll \nu_{el}$ always. Moreover, this correction in all probability

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** Hence if we prescind from all other remarks, the boundary condition $dj_t / dz = 0$ (see reference 13), corresponding to the condition on the free surface of a viscous liquid, may appear justified to one extent or another. For reasons clear from the text we shall not consider this question in greater detail.

lies beyond the limit of exactness to which existing formulas of the theory of the anomalous skin effect may pretend, especially in the region where $l_l \lesssim \delta$. This becomes clear if we recall that the formulas of which we are speaking (for example, the expression for the surface impedance Z) are obtained[16,17,2] on the basis of assumptions concerning the sphericity of the Fermi surface, the specific character of the reflection of the electrons from the metal boundary and the possibility of approximating the integral of the collisions by means of an expression of the type of Eq. (7) even in the vicinity of the boundary, where the distribution function is not a slowly changing function of the angles in the velocity space. Thus in the present state of the question it is not necessary to take account of interelectronic collisions in the theory of the anomalous skin effect. This refers, in particular, even to the far infrared region of the spectrum[‡].

[‡] In view of what has been said, it is clear that the work of reference 1 is incorrect both with respect to the method employed in it and with respect to the results obtained. Moreover, it is necessary to keep in mind that on going over to sufficiently high frequencies, where the condition $\hbar\omega \ll kT$ is not fulfilled, calculations based on the ordinary use of the kinetic equation are already invalid, generally speaking, on account of the necessity of taking account of quantum effects (see reference 19, Sec. 4B). These quantum effects were not taken into account above. The carrying out of a similar calculation using a quantum kinetic equation[20] would be very useful. In the region $\hbar\omega \gtrsim kT$ interelectronic collisions may be essential (see reference 19, Sec. 4B).

[19] V. L. Ginzburg and G. P. Motulevich, Usp. Fiz. Nauk SSSR 55, 469 (1955)

[20] Iu. L. Klimontovich and V. P. Silin, J. Exper. Theoret. Phys. USSR 23, 151 (1952)

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Nucleon-Nucleon Scattering According to the Theory of Damping

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Nucleon-nucleon scattering cross sections have been calculated on the basis of the theory of damping. It is shown that the calculated dependence of the total cross sections on the energy cannot be made to agree with experiment.

IT is well known that theoretical attempts to explain the behavior of nucleon-nucleon scattering cross sections at high energies have not been successful to date. In particular, the examination of this problem within the framework of perturbation theory[1,2] also has not led to satisfactory results insofar as the angular dependence of the proton-proton and proton-neutron scattering cross sections turned out to be in sharp disagreement with experiment. At the same time it was shown that[1] the total scattering cross sections found on the basis of perturbation theory depend only weakly on the energy over a wide energy range and remain approximately constant, in agreement with experimental data. However, inasmuch as the perturbation theory results cannot be trusted, in view of the large value of the nuclear interaction constant, it seeemed to be of methodological interest to investigate the problem of nucleon-nucleon scattering within the framework of the theory of damping, which would enable one formally to deal with large interaction constants and to compare results obtained by different methods. The results of such a calculation are presented below.

2. We write the basic integral equation of the theory of damping in the form[3,4]:

$$R(\mathbf{k}'s_1's_2'\tau_1'\tau_2' \mid \mathbf{k}s_1s_2\tau_1\tau_2) \tag{1}$$
$$= K(\mathbf{k}'s_1's_2'\tau_1'\tau_2' \mid \mathbf{k}s_1s_2\tau_1\tau_2)$$
$$-\frac{ikE_k}{(4\pi)^2}\int d\Omega_q K(\mathbf{k}'s_1's_2'\tau_1'\tau_2' \mid \mathbf{q}s_1''s_2''\tau_1''\tau_2'')$$
$$\times\ R(\mathbf{q}s_1''s_2''\tau_1''\tau_2'' \mid \mathbf{k}s_1s_2\tau_1\tau_2),$$

where R is the scattering amplitude, $K = iS_2$. Here S_2 is the matrix element of the S-matrix[5] quadratic in the interaction constant. We shall consider the case of nucleons interacting through a pseudo-scalar meson field subject to the sum of two couplings: a pseudoscalar coupling characterized by the constant g, and a pseudovector coupling characterized by the constant g'. In this case the quantity K occurring in Eq. (1) may be represented in the form:

$$K(\mathbf{k}'s_1's_2'\tau_1'\tau_2' \mid \mathbf{k}s_1s_2\tau_1\tau_2) \tag{2}$$
$$= \frac{G^2}{(2E_k)^2}\left\{\frac{(\vec{\sigma}\cdot\mathbf{k}-\mathbf{k}')_{s_2's_2}(\vec{\sigma}\cdot\mathbf{k}-\mathbf{k}')_{s_1's_1}\vec{\tau}_{\tau_2'\tau_2}\vec{\tau}_{\tau_1'\tau_1}}{2k^2(1-\cos\vartheta_{kk'})+\mu^2}\right.$$
$$\left.-\frac{(\vec{\sigma}\cdot\mathbf{k}+\mathbf{k}')_{s_1's_2}(\vec{\sigma}\cdot\mathbf{k}+\mathbf{k}')_{s_2's_1}\vec{\tau}_{\tau_1'\tau_2}\vec{\tau}_{\tau_2'\tau_1}}{2k^2(1+\cos\vartheta_{kk'})+\mu^2}\right\}.$$

Here $G = g + (2M/\mu)g'$, σ, τ are Pauli matrices, $\mathbf{k}$ is the momentum of the colliding nucleons in the center of mass system s_1, s_2, τ_1, τ_2 are the spins and isotopic spins of the two colliding nucleons; $\mathbf{k}'$, s_1', s_2', τ_1', τ_2' are the corresponding quantities after scattering, μ is the meson mass, M is the nucleon mass and $E_k = (k^2 + M^2)^{1/2}$.

3. In order to solve the integral equation (1) we first of all separate out the isotopic spin variables by setting

$$R = \Psi^{*T=0}R^{T=0}\Psi^{T=0} + \Psi^{*T=1}R^{T=1}\Psi^{T=1}, \tag{2a}$$

where $\Psi^{T=0}$ and $\Psi^{T=1}$ are the isotopic spin functions of the system of two nucleons which correspond to the total isotopic spin of zero or one, respectively. Following this, Eq. (1) separates into two independent equations which determine the functions $R^{T=0}$ and $R^{T=1}$. We solve the equation for $R^{T=0}$ by expanding the unknown function into a series with respect to the total angular momentum eigenfunctions. Here q denotes the totality of all the angle and spin coordinates $q = \{\Omega, s_1, s_2\}$.

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[1] M. Jean and J. Prentki, J. Phys. et radium 11, 33 (1950)

[2] C. Marty, J. Phys. et radium 12, 833 (1951)

[3] N. Fukuda and T. Miyazima, Prog. Theoret. Phys. 5, 849 (1950)

[4] J. Pirenne, Phys. Rev. 86, 395 (1951)

[5] F. J. Dyson, Phys. Rev. 75, 1736 (1949)

$$R^{T=0}(\mathbf{k}'s_1's_2' \,|\, \mathbf{k}s_1s_2) = \sum \{W_M^{J*}(q)\,A_1 W_M^J(q') \tag{3}$$

$$+ W_M^{J,J*}(q)\,A_2 W_M^{J,J}(q')$$

$$+ W_M^{J,J-1*}(q)\,B_1 W_M^{J,J-1}(q')$$

$$+ W_M^{J,J+1*}(q)\,B_2 W_M^{J,J-1}(q')$$

$$+ W_M^{J,J-1*}(q)\,B_3 W_M^{J,J+1}(q')$$

$$+ W_M^{J,J+1*} B_4 W_M^{J,J+1}(q')\}.$$

The normalized eigenfunctions of the total angular momentum in our case have the form:

$$W_M^J(q) = Y_M^J(\Omega)\,S_0(s_1s_2); \tag{4}$$

$$W_M^{J,J} = \sqrt{\frac{(J+M)(J-M+1)}{2J(J+1)}}\,Y_{M-1}^J\,{}^3S_1$$

$$+ \frac{M}{\sqrt{J(J+1)}}\,Y_M^J\,{}^3S_0$$

$$- \sqrt{\frac{(J+M+1)(J-M)}{2J(J+1)}}\,Y_{M+1}^J\,{}^3S_{-1};$$

$$W_M^{J,J-1} = \sqrt{\frac{(J+M)(J+M-1)}{2J(2J-1)}}\,Y_{M-1}^{J-1}\,{}^3S_1$$

$$- \sqrt{\frac{(J+M)(J-M)}{J(2J-1)}}\,Y_M^{J-1}\,{}^3S_0$$

$$+ \sqrt{\frac{(J-M)(J-M-1)}{2J(2J-1)}}\,Y_{M+1}^{J-1}\,{}^3S_{-1};$$

$$W_M^{J,J+1} = \sqrt{\frac{(J-M+2)(J-M+1)}{(2J+2)(2J+3)}}\,Y_{M-1}^{J+1}\,{}^3S_1$$

$$+ \sqrt{\frac{(J+M+1)(J-M+1)}{(J+1)(2J+3)}}\,Y_M^{J+1}\,{}^3S_0$$

$$+ \sqrt{\frac{(J+M+1)(J+M+2)}{(2J+2)(2J+3)}}\,Y_{M+1}^{J+1}\,{}^3S_{-1};$$

where $S_0(s_1, s_2)$ and ${}^3S_m(s_1, s_2)$ are the singlet and triplet spin functions for a system of two nucleons. The spherical harmonics $Y_m^l(\Omega)$ are defined in accordance with reference 6.

On substituting the expansion (3) into the equation determining $R^{T=0}$, one finds the expansion coefficients A_i and B_i. We shall not give here the explicit form of these coefficients insofar as we shall be interested only in the total nucleon scattering cross section, into which enter only the squares of the absolute values of these coefficients given below.

The total nucleon-nucleon scattering cross section is determined by the formula

$$\sigma = \frac{1}{4}\sum_{\text{spin}} \frac{E_k^2}{(4\pi)^2}\int d\Omega_k\,|R|^2. \tag{5}$$

Substituting the expansion (3) into Eq. (5) we obtain the total nucleon scattering cross section in the state $T = 0$

$$\sigma^{T=0} = \frac{4\pi}{k^2}\sum_{J\geq 1}(2J+1)\left\{\frac{a_1^2 s_0}{1+4a_1^2} + \frac{a_2^2 s_1}{1+4a_2^2}\right. \tag{6}$$

$$\left. + \frac{b_1^2 + b_3^2 + 2b_2^2 + 8(b_1b_3 - b_2^2)^2}{1 + 4(b_1^2 + b_3^2 + 2b_2^2) + 16(b_1b_3 - b_2^2)^2}\,s\right\}.$$

Here the following notation has been used:

$$a_1 = U\left(\frac{J+1}{2J+1}K_{J+1} + \frac{J}{2J+1}K_{J-1} - K_J\right); \tag{7}$$

$$a_2 = U\left(K_J - \frac{J+1}{2J+1}K_{J-1} - \frac{J}{2J+1}K_{J+1}\right);$$

$$b_1 = U\frac{1}{2J+1}(K_{J-1} - K_J);$$

$$b_2 = U\frac{\sqrt{J(J+1)}}{2J+1}(K_{J-1} + K_{J+1} - 2K_J);$$

$$b_3 = U\frac{1}{2J+1}(K_J - K_{J+1});$$

$$U = \frac{k}{E_k}\frac{G^2 t_0}{2^6\pi};\quad K_J = \int_{-1}^{+1}\frac{P_J(x)\,dx}{z - x};\quad z = 1 + \frac{\mu^2}{2k^2};$$

$$s_0 = \frac{1}{2}[1 - (-1)^J];\quad s_1 = \frac{1}{2}[1 + (-1)^J];$$

$$s = \frac{1}{2}[1 + (-1)^{J+1}];\quad t_0 = -3;$$

$P_j(x)$ is the Legendre polynomial normalized by the condition

$$\int_{-1}^{+1} P_J^2(x)\,dx = \frac{2}{2J+1}. \tag{7a}$$

4. The equation for $R^{T=1}$ is solved by a quite analogous method and the following expression is obtained for the total nucleon scattering cross

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[6] H. Bethe, Quantum Mechanics of the Simplest Systems, Handbuch der Physik, vol. XXIV-1

section in the state $T = 1$

$$\sigma^{T=1} = \frac{4\pi}{k^2} \sum_{J>1} (2J+1) \left\{ \frac{a_1^2 s_0}{1+4a_1^2} + \frac{a_2^2 s_1}{1+4a_2^2} + \frac{b_1^2 + b_3^2 + 2b_2^2 + 8(b_1 b_3 - b_2^2)^2}{1 + 4(b_1^2 + b_3^2 + 2b_2^2) + 16(b_1 b_3 - b_2^2)^2} s \right\} + \frac{4\pi}{k^2} \left\{ \frac{a_1^2(J=0)}{1+4a_1^2(J=0)} \right.$$

$$\left. + \frac{b_3^2(J=0) + 4[b_1(J=0)\, b_3(J=0) - b_2^2(J=0)]^2}{1 + 4[b_1^2(J=0) + b_3^2(J=0) + 2b_2^2(J=0)] + 16[b_1(J=0)\, b_3(J=0) - b_2^2(J=0)]^2} \right\}. \tag{8}$$

The quantities a_i, b_i and s_i occurring in formula (8) differ from the corresponding quantities given in Eq. (7) by the fact that in (7) the following substitutions should be made:

$$t_0 \to t_1 = 1; \quad s_0 \to {}^1/_2 [1 + (-1)^J]; \tag{8a}$$

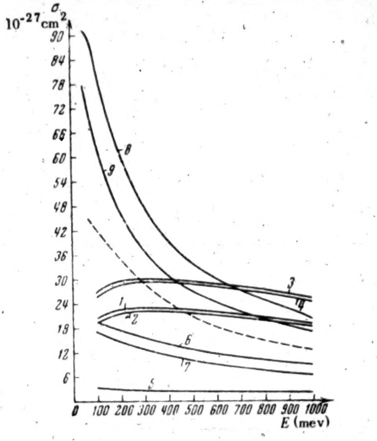
$$s_1 \to {}^1/_2 [1 - (-1)^J];$$

$$s \to {}^1/_2 [1 - (-1)^{J+1}].$$

5. Knowing the cross sections $\sigma^{T=0}$ and $\sigma^{T=1}$, it is easy to obtain the total proton-proton and proton-neutron scattering cross sections with the aid of formulas

$$\sigma_{pp} = \sigma^{T=1}; \quad \sigma_{pn} = {}^1/_4 (\sigma^{T=0} + \sigma^{T=1}). \tag{9}$$

The Figure shows graphs of the dependence of σ_{pp} and σ_{pn} on the energy for various values of the interaction constant $G = g + (2M/\mu) g'$. For comparison, the same Figure also gives curves obtained by neglecting damping, i.e., corresponding to the perturbation theory results*[1]. From the Figure it may be seen that while the total cross sections calculated by perturbation theory for $G^2 \sim 70$ represents quite well the experimentally observed independence of the total cross sections on the energy, the curves which correspond to the more consistent method of calculation on the basis of the theory of damping disagree sharply with the experimental data**. As may be seen from the Figure, for large values of the interaction constant the theoretical curves depend strongly on the energy, while for smaller values of the interaction constant, although a weaker dependence of the cross sections on the energy is obtained, nevertheless, the absolute values of the cross sections turn out to be too small. Therefore, in the case



The dependence on the nucleon energy (in the laboratory system) of the total cross sections σ_{pp} and σ_{pn} for various values of the constsnt G^2. Curves 1-4 correspond to perturbation theory, 5-9 to damping theory. 1. σ_{pn}, $G^2 = 70$; 2. σ_{pp}, $G^2 = 70$; 3. σ_{pn}, $G^2 = 80$; 4. σ_{pp}, $G^2 = 80$; 5. $\sigma_{pp} = \sigma_{pn}$, $G^2 = 80/3$; 6. σ_{pp}, $G^2 = 80$; 7. σ_{pn}, $G^2 = 80$; 8. σ_{pp}, $G^2 = 240$; 9. σ_{pn}, $G^2 = 240$; the dotted curve shows the approximate energy dependence of the cross sections σ_{pp} and σ_{pn} for a value of the constant G^2 intermediate between 80 and 240. The experimental data for σ_{pp} correspond to a cross section which is independent of the energy, and is approximately equal to 24×10^{-27} cm^2.

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* Our Eqs. (9) agree with the corresponding formulas of reference 1 in the case that damping is neglected. However, it should be noted that in reference 1 a factor equal to 1/4 has been left out which corresponds to averaging over the spins of the initial nucleons, in view of which the formulas of reference 1 must be multiplied by 1/4 before the cross sections are compared.

** We also note that at sufficiently high energies, processes of inelastic scattering of nucleons accompanied by production of mesons become possible. We have not taken such processes into account.

of nucleon-nucleon scattering, the theory of damping in the approximation considered cannot be made to agree with experiment, as was also the situation in the case of the scattering of mesons by nucleons[7].

Translated by G. M. Volkoff
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[7] G. F. Zharkov, J. Exper. Theoret. Phys. USSR 27, 296 (1954)

SOVIET PHYSICS JETP VOLUME 2, NUMBER 1 JANUARY, 1956

The Interaction of Fast Deuterons with Nuclei

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In the interaction of fast deuterons with nuclei, the processes usually considered are stripping, deuteron capture, electric dissociation and diffraction scattering of the deuteron as a whole. It is shown that, in addition, as a result of the diffraction of deuterons, dissociation of the deuteron takes place outside the nucleus, the nucleus recoiling only slightly in the process. A general formula is derived which can serve as basis for calculations on diffraction dissociation induced by either purely nuclear forces or by nuclear and electric forces.

1. IN the analysis of the interaction between nuclei and fast, nonrelativistic deuterons (energy $E_0 \sim 30 - 300$ mev), it is usual to consider only three processes: a) capture of the deuteron by the nucleus (followed by a nuclear reaction), b) capture by the nucleus of the neutron (or proton) in the deuteron, with an accompanying nuclear reaction and emission of a fast proton (or neutron) --- the so-called stripping process and c) dissociation of the deuteron into a free proton and neutron under the influence of the electric field of the nucleus.

The first of these processes, deuteron capture, has the largest cross section. For a black nucleus, $\sigma_{\text{capt}} = \pi(R + R_d)^2$, where R and R_d are some effective radii for the nucleus and deuteron. For medium and heavy nuclei, the stripping cross section is less by a factor of order $R_d/(R + R_d)$ (for a black nucleus, $\sigma_{\text{st}} \approx \pi/2RR_d)$[1]. The cross section σ_{el} for dissociation in an electric field is still smaller, except for the heaviest nuclei (this cross section cannot be expressed by such a simple formula).

To this list we should add d) diffraction scattering of the deuteron as a whole --- a process which at the present time can be considered trivial. It is a necessary consequence of the existence of all the other processes depleting the initial deuteron beam, and has a cross section of order of magnitude the same as the largest indicated above, i.e., $\sigma^{\text{diff}}_{\text{scatt}} \sim \pi(R + R_d)^2$ (for a black nucleus).

However, there is yet another process, of very unusual character, and with a cross section, as we shall show, comparable to the cross sections of processes b) and c). This process is associated with the fact that in the diffraction scattering of deuterons, a certain momentum is given to the nucleus as a whole. As a consequence, not only can the deuteron be scattered, but it may also dissociate and give rise to a free proton and neutron. The characteristic feature of this diffraction dissociation is that the nucleus absorbs the momentum as a whole, and undergoes no nuclear reaction. Similar physical phenomena have been investigated in several recent papers on the theory of diffraction processes[2,3]. These phenomena are characterized by the fact that when the colliding particles have large energies, the momentum transfer in the direction of motion becomes small, and hence, according to the uncertainty principle, the volume in which the process takes place becomes very large (and can be much larger than the nucleus). Thus, diffraction dissociation can be called a process of external dissociation. The probabilities for the process to occur in various nuclei are indicated at the end of reference 3. Its physical basis is the same as that in the emission of gamma rays by fast charged π-mesons[2] and of π-mesons by fast nucleons[3].

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[1] A. I. Akhiezer and I. Ia. Pomeranchuk, *Some Problems in the Theory of the Nucleus*, Second Edition, Moscow 1952, Secs. 13, 14

[2] L. D. Landau and I. Ia. Pomeranchuk, J. Exper. Theoret. Phys. USSR 24, 505 (1953); I. Ia. Pomeranchuk, Dokl. Akad. Nauk SSSR 96, 265 (1954)

[3] I. Ia. Pomeranchuk and E. L. Feinberg, Dokl. Akad. Nauk SSSR 93, 439 (1953)

2. The following three conditions must be satisfied if diffraction dissociation is to occur: a) the deuteron must receive an impulse of the order of magnitude of the reciprocal of its dimensions, $q \sim 1/R_d \sim 0.8\ \mu$; b) the volume in which the process takes place must be large in at least one of its dimensions (the longitudinal one) compared to the nuclear radius R, i.e., the dissociation is associated with a momentum transfer in the direction of motion which satisfies the relation $q_{\|} R \ll 1$; c) the deuteron must receive the necessary impulse in a sufficiently short time, having an order of magnitude of the inverse of the binding in the deuteron (in energy units, the deuteron binding energy).

A large fraction of the diffraction scattering is at an angle of approximately $\lambdabar/R$, where $\lambdabar = 1/p_d$ and p_d is the momentum of the deuteron as a whole. Hence,

$$q \sim p_d \lambdabar/R \sim 1/R. \tag{2.1}$$

In our calculations (see below), we shall consider this to be considerably smaller than $1/R_d$. Hence, only a small fraction of the scattering is effective in the sense of dissociation, and the dissociation cross section is much smaller than the cross section for diffraction scattering. One can easily convince oneself that in this case it is approximately RR_d, although the order of magnitude should be given by this formula even for $R \sim R_d$. Here the momentum transferred is small compared to the momentum of the emitted particles. It follows that the distributions in energy and angle of the emitted particles are given essentially by the momentum distribution inside the deuteron. In particular, the momenta of the proton and neutron upon dissociation are almost equal. In general, the distributions must be similar to those in the stripping reaction.

In order to check whether conditions b) and c) are satisifed, we must calculate $q_{\|}$.

Let p_d, p_n and p_b be the momenta of the deuteron before dissociation of the emitted neutron and of the proton respectively, and ϑ_{nd}, ϑ_{pd} be the angles (with respect to the direction of the incident deuteron beam) at which the neutron and proton are emitted; then

$$\begin{aligned} q_{\|} &= p_d - p_n \cos\vartheta_{nd} - p_p \cos\vartheta_{pd} \\ &= p_d - p_n - p_p + p_n(1-\cos\vartheta_{nd}) \\ &\quad + p_p(1-\cos\vartheta_{pd}). \end{aligned} \tag{2.2}$$

Let M be the mass of a nucleon and ϵ the deuteron binding energy. Then, for sufficiently large deuteron energies, namely,

$$E_d \approx p_d^2/4M \gg \varepsilon$$

conservation of energy,

$$\frac{p_d^2}{2(2M-\varepsilon)} - \varepsilon = \frac{p_n^2}{2M} + \frac{p_p^2}{2M}$$

(where we neglect the recoil energy of the nucleus, see below), leads to the formula

$$\begin{aligned} p_d &\approx \sqrt{2}\sqrt{p_n^2 + p_p^2} + \frac{2M\varepsilon}{p_d} \\ &\approx \sqrt{2}\sqrt{p_n^2 + p_p^2}\left(1 + \frac{M\varepsilon}{p_n^2 + p_p^2}\right). \end{aligned} \tag{2.3}$$

For small angles ($\vartheta_{nd} \approx \vartheta_{pd} = 0$), which imply a small value for $q_{\|} = q_{\|\min}$, we obtain the following conditions for the possibility of external dissociation: (R is the nuclear radius, and as always, $h = c = 1$):

$$\begin{aligned} q_{\|\min} R = \Big\{ &\sqrt{2}(p_n^2 + p_p^2)^{1/2} - p_n \\ &- p_p + \frac{\sqrt{2}M\varepsilon}{\sqrt{p_n^2 + p_p^2}} \Big\} R \ll 1. \end{aligned} \tag{2.4}$$

Since, as noted above, $p_n \approx p_p \approx 1/2 p_d$ (more exactly, it is necessary that $|p_n - p_p| \lesssim \sqrt{4M\epsilon}$; subsequent calculations will confirm that this gives the most significant range of p_n, p_p), the condition (2.4) becomes:

$$q_{\|} R = (2M\varepsilon/p_d) R \ll 1, \tag{2.4a}$$

or

$$\begin{aligned} \frac{p_d^2}{4M} &\approx E_d \gg R^2 M \varepsilon^2 \\ &= A^{2/3} \frac{M}{\mu} \frac{\varepsilon}{\mu c^2} \varepsilon \sim 0.2\, A^{2/3}\ \text{MeV} \end{aligned} \tag{2.5}$$

(where for convenience we put $R = A^{1/3}\, 1/\mu$).

Hence for deuterons with energy in the tens of mev, condition b) is satisfied.

It will be shown later that the effective angles are approximately $\vartheta_{nd}^2 \sim \vartheta_{np}^2 \sim 4M\epsilon/p_d^2 \sim q_{\|\min}/p_d^2$, so that the above holds also for all angles which occur.

Finally, to check whether condition c) holds, we consider the collision time τ. This is evidently the time of flight of a deuteron through a length of order $1/q_{\|}$, i.e., from (2.4a):

$$\tau \sim \frac{1}{q_{\|}} \frac{2M}{p_d} \sim \frac{1}{\epsilon}. \tag{2.6}$$

Hence, the third condition is also satisfied. We note that at relativistic energies, a calculation of $q_{\|}$ using energy conservation as in (2.1) - (2.4a) leads to a similar result, namely, that in the rest frame of the deuteron, the momentum is transferred in time $1/\epsilon$. This leads to the supposition that the process in question should also go at relativistic energies, and be of similar character.

3. The basic general formula for the cross section of the process being considered can be obtained in several different ways. We shall use the Born approximation in the sense that the interaction U between the emitted proton and neutron and with the electric field W of the nucleus will be considered to be small perturbations. The nucleus will appear as a fixed source of a nuclear potential V acting on the proton and neutron of the deuteron. We neglect exchange effects between these particles and the nuclear protons and neutrons (this is permissable since the energy of the deuteron's nucleons is large compared to the energy of the nuclear nucleons).

Let us, for example, use the formal apparatus and notation of the scattering theory described in reference 4. Let the full energy operator for the system nucleus plus deuteron be written in the form:

$$H = K + U + V + W. \tag{3.1}$$

Here K is the sum of the operators for the deuteron kinetic energy and the internal energy of motion in the nucleus.

We are interested in the following wave functions:

First, the exact wave function for the system, satisfying the initial condition --- a free incident deuteron of energy E_a --- and containing as reaction products only outgoing waves (from the nucleus) $\Psi^{+\epsilon}$. If the state function: incident deuteron plus unperturbed nucleus is denoted by Ψ_{0a}, then evidently $\Psi_a^{(+\epsilon)} = \Psi_{0a}$

$$+ \frac{1}{E_a - K - U + i\epsilon} (V + W) \Psi_a^{(+\epsilon)}, \tag{3.2}$$

$$(E_a - K - U) \Psi_{0a} = 0. \tag{3.2a}$$

In the absence of a nuclear field, $V = W = 0$, and this expression reduces to Ψ_{0a} as it should; for $\epsilon = 0$ it satisfies the equation $H\Psi_a^{(+0)} = E_a \Psi_a^{(+0)}$ while the term $+i\epsilon$, which will go to zero at the end of the calculation. corresponds to outgoing waves. We write $\Psi_a^{(+0)} = e^{-iE_a t} \psi_a^{(+)}$.

Second, the wave function φ_b for the experimentally observed situation, i.e., a free neutron and proton of definite momenta p_n and p_p (product of two plane waves). Evidently

$$(K - E_b) \varphi_b = 0. \tag{3.3}$$

Third, and last, the function $\chi_b^{(-)}$ describing a neutron and proton of momenta p_n and p_p interacting with the nuclear field V, but not with each other, nor with the electric field of the nucleus. We can write

$$\varphi_b = \chi_b^{(-)} - \frac{1}{E_b - K - i\epsilon} V \chi_b^{(-)}. \tag{3.4}$$

The $\chi_b^{(-)}$ determined in this way reduces to φ_b for $V = 0$, as it should, and for $\epsilon = 0$ it satisfies the equation $(K + V - E_b)\chi_b^{(-)} = 0$. The term $-i\epsilon$ (note the minus sign!) corresponds to an incoming wave in the space co-ordinate, as is necessary to describe the production of particles with a definite momentum.

Given that the full solution is $\Psi^{(+\epsilon)}$, we are interested in the probability of finding the state φ_b, i.e.,

$$w_{ba} = |\langle \varphi_b \exp\{-iE_b t\}| \tag{3.5}$$

$$\times \exp\{-i(K + U + V + W)t\} \,|\, \psi_a^{(+)} \rangle|^2.$$

From this it can be easily calculated[4] that the transition probability per unit time at $t = 0$ is

$$w_{ba}|_{t=0} = 2\pi |\langle \varphi_b | U + V \tag{3.6}$$

$$+ W | \psi_a^{(+)} \rangle|^2 \delta(E_a - E_b).$$

Using Eq. (3.4), we have

$$\langle \varphi_b | U + V + W | \psi_a^{(+)} \rangle \tag{3.7}$$

$$= \langle \chi_b^{(-)} | U + V + W | \psi_a^{(+)} \rangle$$

$$- \langle \frac{1}{E_b - K - i\epsilon} V \chi_b^{(-)} | U + V + W | \psi_a^{(+)} \rangle$$

$$= \langle \chi_b^{(-)} | U + V + W | \psi_a^{(+)} \rangle$$

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[4] M. Gell-man and M. Goldberger, Phys. Rev. **91**, 398 (1953)

$$-\left\langle \chi_b^{(-)} \middle| V \frac{1}{E_b - K + i\varepsilon} (U + V + W) \middle| \psi_a^{(+)} \right\rangle.$$

But according to Eq. (3.2),

$$(E_a + i\varepsilon - K)\,\psi_a^{(+\varepsilon)}$$

$$= (U + V + W)\,\psi_a^{(+\varepsilon)} + i\varepsilon\psi_{0a},$$

so that for $\epsilon \to 0$ we get

$$\frac{1}{E_a - K + i\varepsilon}(U + V + W)\,\psi_a^{(+\varepsilon)} \to \psi_a^{(+)},$$

and hence,

$$w_{ba}|_{t=0} = 2\pi \,|\langle \chi_b^{(-)} |\, U \qquad (3.8)$$

$$+ W \,|\, \psi_a^{(+)} \rangle|^2\, \delta(E_a - E_b).$$

So far, this formula is exact. Let us consider now the function $\psi_a^{(+1)}$. It is the superposition of the following four states: a) incident plane waves and outgoing diffracted waves for the deuteron as a whole, $\psi_a + \psi_d^{\text{diff}}$; b) emitted proton plus neutron captured by the nucleus, ψ_{prot}; c) emitted neutron and captured proton, ψ_{neut}; d) the state of interest to us, i.e., simultaneously emitted proton and neutron (strictly speaking, interacting with each other, and with the electric and nuclear fields of the nucleus ---ψ_f^{diff}

By Born's approximation we mean the substitution of $\psi_a + \psi_d^{\text{diff}}$ for $\psi_a^{(+)}$ in Eq. (3.8). Our neglect of the other terms is justified as follows: With ψ_d^{diff} we take into account a term which leads to diffraction scattering of the deuteron with cross section $\sim \pi(R + R_d)^2$. The last constituent (d describes diffraction dissociation which is associated with the transfer to the deuteron of a transverse momentum of order $1/R_d$. As in other processes of this kind[2,3], we expect (see above) it to have a cross section $\sim RR_d$ (this will be confirmed by a later calculation). Hence, our approximation is applicable to nuclei sufficiently heavy that

$$R_d/(R + R_d) \ll 1. \qquad (3.9)$$

The stripping process also has a cross section $\sim RR_d$, and on the same grounds we could neglect the functions b) and c). Actually, the situation here is even better (at least neglecting the effect of the electric field). The point here is that the interaction U is almost a δ-function, so that when one of the particles is captured by the nucleus, and its function is concentrated inside the nucleus, then it cannot interact with particles in the state $\chi^{(-)}$, which penetrate only weakly into the nucleus. For these reasons, we can calculate from the formula

$$w_{ba} = 2\pi \,|\langle \chi_b^{(-)} |\, U \qquad (3.10)$$

$$+ W \,|\, \psi_0 + \psi_d^{\text{diff}} \rangle|^2\, \delta(E_a - E_b).$$

This formula, with $W = 0$, can be obtained by another method[5] which was used to study the purely electrical dissociation of the deuteron. We need only replace the electric potential of the nucleus by V and write the deuteron function in the form $\psi_{0a} + \psi_d^{\text{diff}}$. However, in this method the nature of the approximations made is not clear.

The function ψ_d^{diff} may be found either for deuteron wavelengths small compared to R, or for a black nucleus, in which latter case the derivation is particularly simple.

We now justify the Born approximation for the proton-neutron interaction in the final state. Since the particles separate with a relative momentum of the same order of magnitude as their relative momentum inside the deuteron, i.e., $p_n - p_p \sim 2\sqrt{M\epsilon}$, the kinetic energy of this motion is of the order $1/M(p_n - p_p)^2 \sim 4\epsilon \sim 8.5$ mev. But it is well-known that in this energy range the proton-neutron scattering can be described in Born's approximation with the help of a certain pseudo-potential, whose absolute value, it is true, depends on the energy. In this way, if U is regarded as a certain effective operator for the interaction (if necessary, an integral operator), then the Born approximation will be applicable. The subsequent calculations will, in some cases, give the explicit form of this potential. Thus we see that even if we neglect the electric field W of the nucleus, the phenomenon of diffraction dissociation must exist; it will depend on diffraction of the deuteron (the term ψ_d^{diff}), and of the reaction products (if $\chi_b^{(-)}$ is replaced by the product of plane waves and ψ_d^{diff} is discarded, then conservation laws will make w_{ba} zero).

The theory of the electric dissociation of fast deuterons[6] would correspond in Eq. (3.10) not only to neglecting U, but also to replacing $\chi_b^{(-)}$ by plane waves φ_b, and discarding ψ_d^{diff}. Thus, we see that the effect of diffraction on purely electrical dissociation can also be taken into account. Hence, working from Eq. (3.10), the theory of electric dissociation can be improved.

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[5] L. D. Landau and E. M. Lifschitz, J. Exper. Theoret. Phys. USSR 18, 750 (1948) (see reference 1, Sec. 13)

[6] S. M. Dancoff, Phys. Rev. 72, 1017 (1947) (see reference 1, Sec. 13)

Since in order to induce dissociation of the deuteron, it is sufficient to give it an impulse of the order of the reciprocal of its dimensions, the nucleus, as it receives the same impulse, does not absorb an appreciable amount of energy and undergoes no reaction.

The detailed results will be published separately.

Translated by R. Krotkov
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The Connection between the Vibrations of the Surface of a Nucleus and Single Nucleon Excitation

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The conditions of validity of a model of single nucleon excitations in a nucleus are investigated by the method of adiabatic approximation. The effect of the relation between single nucleon excitations and the vibrations of the surface of a nucleus on the excited states of the whole nucleus are established.

1. INTRODUCTION

BECAUSE of strong interaction between the nucleons in a nucleus, one can speak in a strictly defined sense only of a state of a nucleus as a whole and not of the states of an individual nucleon. However, such consideration is still impracticable and one has to apply the approximate methods of study of energy states of nuclei.

In the case of an approximate examination of the lowest energy states, one usually proceeds from the notion that the nucleons in a nucleus are in the form of "shells" which can change their shape and dimensions. Individual nucleons are moving in an average field of the surrounding nucleons. This average field is such that the resultant force acting on one nucleon differs from zero mainly on the surface of a nucleus. Since the average field is caused by many nucleons, its change is connected with the collective movement of the nucleons. Because of the small compressibility of nuclear matter, its density can be considered as being constant. In this approximation, one can picture the "collective" movement only as deformation of nucleus surface without change of the volume.

In a series of cases the frequencies of the collective movements are smaller than the frequencies corresponding to the excitations of individual nucleons in a nucleus. Then, by investigation of the energy states of a nucleus, one can apply the adiabatic approximation.

In this article we will consider the limits of applicability of adiabatic approximation to a nucleus and make clear what effect the connection between the single nucleonic excitations and vibrations of the nuclear surface has on the energy states of a whole nucleus.

Particularly, it will be shown that the probability of a single nucleonic transition under the influence of an external excitation decreases because of a connection between single nuclear states and collective vibrations.

2. THE STRUCTURE OF THE ENERGY SPECTRUM OF A NUCLEUS AT SMALL EXCITATIONS

Let r denote the coordinates and the spins of nucleons in a nucleus and R the configuration of the nuclear surface. Assume that for every value of R the characteristic functions $\varphi_n(r, R)$ and the energy of the nucleons $E_n(R)$ are known. Let the index n denote the set of quantum numbers $\{n, j, m_j\}$, which characterizes the state of all nucleons in a nucleus in the case of a single particle approximation. The functions $\varphi_n(r, R)$ satisfy the equation

$$\{H(r, R) - E_n(R)\}\varphi_n(r, R) = 0. \tag{1}$$

To take into account the change of the shape of the surface of a nucleus, we introduce the kinetic energy operator connected with the change of the nuclear surface:

$$\hat{T}_R \equiv \sum_{\mu} A_\mu \nabla_\mu^2; \tag{2}$$

The explicit expressions for A_μ and ∇_μ will be given later.

The complete wave function, which defines the stationary states of a nucleus, has to satisfy the equation

$$\{\hat{T}_R + H(r, R) - E\}\Psi(r, R) = 0. \tag{3}$$

The solution of Eq. (3) can be sought in the form

$$\Psi(r, R) = \sum_n \varphi_n(r, R)\Phi(n, R), \tag{4}$$

where the $\varphi_n(r, R)$ are the solutions of Eq. (1). By substitution of Eq. (4) into Eq. (2) we get, after simple mathematical manipulations,

$$\begin{aligned}\{\hat{T}_R + E_n(R) - E\}\Phi(n, R)& \\ = \sum_m D_{nm}\Phi(m, R).&\end{aligned} \tag{5}$$

The operator D_{nm} is determined by the expression

$$\begin{aligned}D_{nm} &= \int \varphi_m^* \hat{T}_R \varphi_n (dr) \\ &\quad + 2\sum_\mu A_\mu \int \varphi_m^* \nabla_\mu \varphi_n (dr) \nabla_\mu.\end{aligned} \tag{6}$$

The presence of non-diagonal terms on the right-hand side of Eq. (5) indicates that the state $\varphi_n(r, R)\Phi(n, R)$ is not stationary. If at some instance this state is fixed, then after some time spontaneous transitions into other states with the same energy will take place.

When the non-diagonal terms on the right-hand side of Eq. (5) are small, (we will evaluate these terms later) then in the zeroth approximation we will get the system of equations

$$\{T_R + E_n(R) - E_\nu\}\Phi_\nu(n, R) = 0, \tag{7}$$

where $E_n(R)$ plays the part of the potential energy of surface deformation. Equation (7) characterizes the vibrations of the nuclear surface about the equilibrium positions determined (for each state of n-particle model) from the condition of minimum of $E_n(R)$. Let n_0 denote the complete set of quantum numbers $\{n, j, m_j\}$ for which the nucleons have the lowest of all possible energy states satisfying the Pauli principle. The energy of this state $E_{n_0}(R)$ will be a function of configuration R. The equilibrium configuration of the nucleus will be determined by the condition min $E_{n_0}(R)$. We expand $E_n(R)$ in powers of relative departure $\alpha = (R - R_0)/R_0$ from a spherical shape of the nucleus R_0:

$$E_n(R) = E_n(R_0) + V_n^{(1)}\alpha + V_n^{(2)}\alpha^2 + \dots \tag{8}$$

The term $V_n^{(2)}\alpha^2$ can be considered as the potential energy of a surface deformation and can be expressed in terms of "macroscopic" parameters, for instance, the surface tension γ. Let

$$V_n^{(2)}\alpha^2 \equiv {}^1\!/_2 \sum_{\lambda\mu} C_\lambda^{(n)} |\alpha_{\lambda\mu}|^2, \tag{9}$$

where $\alpha_{\lambda\mu}$ is determined by the expansion in spherical functions of the relative displacement

$$\alpha = \sum_{\lambda\mu} \alpha_{\lambda\mu} \mathrm{Y}_{\lambda\mu}(\vartheta, \phi),$$

where

$$\alpha_{\lambda\mu} = (-)^\mu \alpha_{\lambda,-\mu}^*, \quad \mathrm{Y}_{\lambda\mu}^* = (-)^\mu \mathrm{Y}_{\lambda,-\mu}.$$

The value $\lambda = 1$ corresponds to a displacement of the whole nucleus and not to its deformation. Therefore, in Eq. (9), $\lambda > 1$. The sequence of numbers λ is bounded above by the value $\lambda \approx 6$ because of the finite number of particles in the nucleus. The coefficients $C_\lambda^{(n)}$ depend on quantum states of nucleons. If the distribution of charge is homogeneous in the volume of the spherical nucleus with radius R_0, the $C_\lambda^{(n)}$ are given by[1]

$$C_\lambda^{(n)} = (\lambda - 1)(\lambda + 2)R_0^2\gamma - \frac{3(\lambda - 1)}{2\pi(2\lambda + 1)}\frac{Z^2e^2}{R_0}.$$

To find the expression for the operator of the kinetic energy $\hat{T}_R$, we first consider the classical expression for kinetic energy of surface deformation of a nucleus. The kinetic energy of surface deformation of a nucleus can be expressed in terms of the impulses $\pi_{\lambda\mu} = B_\lambda \dot{\alpha}_{\lambda\mu}^*$ by the expression

$$T_R = \frac{1}{2}\sum_{\lambda\mu} \frac{|\pi_{\lambda\mu}|^2}{B_\lambda}.$$

For an incompressible nucleus with density ρ,

$$B_\lambda = \rho(R_0^5/\lambda).$$

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[1] A. Bohr, Dan. Math. Fys. Medd. 26, 14 (1952); 26, 16 (1953)

By letting

$$\omega_\lambda^{(n)} = \left(\frac{C_\lambda^{(n)}}{B_\lambda}\right)^{1/2},$$

$$\alpha_{\lambda\mu} = \sqrt{\frac{\hbar}{2B_\lambda\omega_\lambda}}\,(b_{\lambda\mu} + (-)^\mu b^{*}_{\lambda,\,-\mu}),$$

$$\pi_{\lambda\mu} = i\sqrt{\frac{\hbar B_\lambda\omega_\lambda}{2}}\,(b^{*}_{\lambda\mu} + (-)^\mu b_{\lambda,\,-\mu}),$$

we get

$$T_R + V_n^{(2)}\alpha^2 \equiv \frac{\hbar}{2}\sum_{\lambda\mu}\omega_\lambda(b_{\lambda\mu}b^{*}_{\lambda\mu} + b^{*}_{\lambda\mu}b_{\lambda\mu}).$$

To go over to quantum mechanics, we will consider that $b_{\lambda\mu}$ are operators satisfying the permutational relationship $[b_{\lambda\mu}, b^{*}_{\lambda'\mu'}] = \delta_{\lambda\lambda'}\delta_{\mu\mu'}$. To simplify the future notation, we will write only one index μ instead of both of them (λ and μ). If new real variables ξ_μ are introduced according to the expressions

$$b_\mu = 2^{-1/2}\left(\xi_\mu + \frac{\partial}{\partial\xi_\mu}\right), \qquad b^{*}_\mu = 2^{-1/2}\left(\xi_\mu - \frac{\partial}{\partial\xi_\mu}\right),$$

we will get

$$\hat{T}_R + V_n^{(2)}\alpha^2 \equiv \frac{\hbar}{2}\sum_\mu \omega_\mu^{(n)}\left\{\xi_\mu^2 - \frac{\partial^2}{\partial\xi_\mu^2}\right\}.$$

The term $V_n^{(1)}\alpha$ in Eq. (8) characterizes the difference between the equilibrium shape of a nucleus in the nth state and the spherical shape. By letting

$$V_n^{(1)}\alpha = -\sum_\mu \hbar\omega_\mu^{(n)}\zeta_\mu^{(n)}\xi_\mu, \tag{11}$$

the surface deformation can be expressed in terms of displacements $\zeta_\mu^{(n)}$ from equilibrium values of variables ξ_μ. Now one can write

$$\hat{T}_R + E_n(R) - E_n(R_0) \tag{12}$$

$$= \frac{\hbar}{2}\sum_\mu \omega_\mu^{(n)}\left\{(\xi_\mu - \zeta_\mu^{(n)})^2 - \frac{\partial^2}{\partial\xi_\mu^2}\right\} - \frac{\hbar}{2}\sum_\mu \omega_\mu^{(n)}\zeta_\mu^2 .$$

By substitution of Eq. (8) into Eq. (7) and taking into consideration Eq. (12), we get the equation which determines the states of vibration of the nuclear surface when the single nucleonic states are characterized by the set of quantum numbers n

$$\left\{\frac{\hbar}{2}\sum_\mu \omega_\mu^{(n)}\left[(\xi_\mu - \zeta_\mu^{(n)})^2 - \frac{\partial^2}{\partial\xi_\mu^2}\right] - \varepsilon_\nu\right\} \tag{13}$$

$$\times\ \Phi_\nu(n; \ldots\xi_\mu\ldots) = 0,$$

where

$$\varepsilon_\nu = E_\nu - E_n(R_0) + \frac{\hbar}{2}\sum_\mu \omega_\mu^{(n)}[\zeta_\mu^{(n)}]^2 . \tag{14}$$

Let ν denote the set of quantum numbers which characterize the vibrational state of a nuclear surface. Equation (13) has the solution

$$\Phi_\nu(n; \ldots\xi_\mu\ldots) = \prod_\mu \psi_{\nu_\mu}(\xi - \zeta_\mu^{(n)}), \tag{15}$$

where $\psi_{\nu\mu}(x)$ is the wave function of a harmonic oscillator. The energy of vibrations corresponding to a certain set of quantum numbers $\nu \equiv (\ldots\nu_\mu\ldots)$, where $\nu_\mu = 0, 1, 2 \ . \ . \ .$ is equal to

$$\varepsilon_\nu^{(n)} = \hbar\sum_\mu \omega_\mu^{(n)}(\nu_\mu + 1/2).$$

The total energy of a nucleus is determined by the quantum states of the nucleons (in a single particle approximation) and by the quantum states of the vibration of a nuclear surface.

$$E_{n\nu} = E_n(R_0) - \frac{\hbar}{2}\sum_\mu \omega_\mu^{(n)}[\zeta^{(n)}]^2 \tag{16}$$

$$+ \hbar\sum_\mu \omega_\mu(\nu_\mu + 1/2).$$

The wave function of such a state is equal to

$$\Psi_{n\nu} = \varphi_n(r, R)\,\Phi_\nu(n; \ldots\xi_\mu\ldots). \tag{15a}$$

In general, the series of states with different n and ν can have approximately equal energies $E_{n\nu} \approx E_{n'\nu'} \approx E_{n''\nu''} \approx \ . \ . \ . \ .$ The states (16) are not stationary. For investigation of the spontaneous transitions of a system from the state $n\nu$ into the state $n'\nu'$ (with approximately the same energy) it is necessary to consider the time equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \{H(r, R) + \hat{T}_R\}\,\Psi. \tag{17}$$

We will seek the solution of this equation in the form $\Psi = \sum_{n\nu} a_{n\nu}(t)\,\Psi_{n\nu}(r, R)$. We then get

$$i\hbar\dot{a}_{n\nu} = E_{n\nu}a_{n\nu} + \sum_{n'\nu'} a_{n'\nu'}F_{n\nu}^{n'\nu'}, \tag{18}$$

where

$$F_{n\nu}^{n'\nu'} = \int \Phi^{*}_{n\nu}D_{nn'}\Phi_{n'\nu'}\,dR. \tag{18a}$$

Assume that at $t = 0$, the nucleus is in the state $n\nu$, i.e., $a_{n'\nu'} = \delta_{nn'}\delta_{\nu\nu'}$. Then the system of equations (18) can be written as follows:

$$i\dot{a}_{n\nu} = \omega_{n\nu}a_{n\nu} + \frac{1}{\hbar}\sum_{n'\nu'} a_{n'\nu'}F^{n'\nu'}_{n\nu},$$

$$i\dot{a}_{n'\nu'} = \omega_{n'\nu'}a_{n'\nu'} + \frac{1}{\hbar} a_{n\nu}F^{n\nu}_{n'\nu'},$$

where $\omega_{n\nu} = E_{n\nu}/\hbar$. By means of the Laplace's transform

$$A_{n\nu} = \int_0^\infty e^{-(\eta - i\gamma)t} a_{n\nu}(t)\,dt,$$

this system of equations can be transformed into the algebraic system

$$A_{n\nu}(\omega_{n\nu} - \gamma - i\eta) = -i - \frac{1}{\hbar}\sum_{n'\nu'} A_{n'\nu'}F^{n'\nu'}_{n\nu},$$

$$A_{n'\nu'}(\omega_{n'\nu'} - \gamma - i\eta) = -\frac{1}{\hbar}\sum_{n\nu} A_{n\nu}F^{n\nu}_{n'\nu'}.$$

By substitution of the second equation into the first one we get

$$A_{n\nu} = \frac{i}{\omega_{n\nu} - \gamma - i[\eta + {}^1/_2 T(\gamma)]},$$

where

$${}^1/_2 T(\gamma) = \frac{1}{i\hbar}\sum_{n\nu} |F^{n'\nu'}_{n\nu}|^2 / (\omega_{n'\nu'} - \gamma - i\eta).$$

By means of an inverse Laplace transform we find

$$a_{n\nu} = \exp\{-i\omega_{n\nu}t - {}^1/_2 T(\gamma_0)t\};$$

where

$$\gamma = \omega_{n\nu} - i\left(\eta + \frac{T(\gamma_0)}{2}\right).$$

Hence the probability of a nucleus to be in the state $n\nu$ varies according to the law $|a_{n\nu}|^2 = e^{-\Gamma t}$, where Γ can be found from the equation

$$1 = \frac{1}{\hbar^2}\sum_{n'\nu'} |F^{n'\nu'}_{n\nu}|^2 / [(\omega_{n'\nu'} - \omega_{n\nu})^2 + {}^1/_4\Gamma^2]. \quad (19)$$

Thus all energy levels at $n \neq n_0$ have the width

$$\Delta E_{n\nu} = \hbar\Gamma, \quad (20)$$

where $1/\Gamma$ is the average lifetime of the state $n\nu$ with respect to a spontaneous transition into some other states.

The adiabatic approximation is valid when the inequality $\hbar\,\Gamma < |E_{n\nu} - E_{n'\nu'}|$ holds. In this case, the terms on the right-hand side of Eq. (19) have sharp maxima for $\omega_{n\nu} = \omega_{n'\nu'}$. Since we assume that the energy of an individual quantum of the surface vibrations is smaller than the excitation energy of a single nucleonic state, the equality $\omega_{n\nu} = \omega_{n'\nu'}$ is possible only when the vibrative state ν' differs from ν by many quanta of vibrations of a nuclear surface.

3. THE CALCULATION OF THE PROBABILITY OF A NONRADIATIVE TRANSITION OF ENERGY OF A SINGLE NUCLEONIC EXCITATION INTO THE VIBRATION ENERGY OF A NUCLEAR SURFACE

The operators $D_{nn'}$, which are functions of R, appear in the matrix element $F^{n'\nu'}_{n\nu}$. To simplify the evaluation of $F^{n'\nu'}_{n\nu}$ we will neglect the less essential first integral $D^*_{nn'}$, which contains the second derivatives of φ_n with respect to ξ_μ. Also, in remaining terms we will replace R by the equilibrium value R_0. Then

$$D_{nn'} = -\hbar\sum_\mu \omega_\mu^{n_0} L_{nn'}(\mu)\frac{\partial}{\partial x_\mu}, \quad (21)$$

where

$$L_{nn'}(\mu) = \left\{\int \varphi^*_{n'}\frac{\partial\varphi_n}{\partial\xi_\mu}(dr)\right\}_{R=R_0} \quad (21a)$$

By the substitution of Eq. (21) into Eq. (18a), and by taking Eq. (15) into consideration, and also the equality

$$\frac{\partial}{\partial\xi_\mu}\Psi_{\nu_\mu} = \sqrt{\frac{\nu_\mu}{2}}\Psi_{\nu_\mu - 1} - \sqrt{\frac{\nu_\mu + 1}{2}}\Psi_{\nu_\mu + 1},$$

we will get

$$F^{n'\nu'}_{n\nu} \approx -\hbar\sum_\mu \omega_\mu^{n_0} L_{nn'}(\mu) \quad (22)$$

$$\times \left\{\sqrt{\frac{\nu_\mu}{2}} M^{nn'}_{\nu_\mu\nu_\mu - 1}\prod_{\mu' \neq \mu} M^{nn'}_{\nu_{\mu'}\nu'_{\mu'}} - \sqrt{\frac{\nu_\mu + 1}{2}} M^{nn'}_{\nu_\mu\nu_\mu + 1}\sum_{\mu' \neq \mu} M^{nn'}_{\nu_{\mu'}\nu'_{\mu'}}\right\},$$

where

$$M^{nn'}_{\nu_\mu\nu'_\mu} = \int \varphi_{\nu_\mu}(\xi_\mu - \zeta_\mu^{(n)})\,\varphi_{\nu'_\mu}(\xi_\mu - \zeta_\mu^{(n')})\,d\xi_\mu. \quad (22a)$$

With the accuracy to the terms of order $(\zeta^n_\mu - \zeta^{n'}_\mu)^2$ the matrix elements $M^{nn'}_{\nu_\mu\nu'_\mu}$ are equal to zero when $\nu'_\mu \neq \nu_\mu$, $\nu_\mu \pm 1$. If $\nu'_\mu = \nu_\mu$, $\nu_\mu \pm 1$, we have

$$M^{nn'}_{\nu_\mu\nu_\mu} = 1 - {}^1/_2(\nu_\mu + {}^1/_2)(\zeta^n_\mu - \zeta^{n'}_\mu)^2; \quad (22b)$$

$$M^{nn'}_{\nu_\mu\nu_\mu + 1} = \sqrt{\frac{\nu_\mu + 1}{2}}(\zeta^n_\mu - \zeta^{n'}_\mu);$$

$$M^{nn'}_{\nu_\mu \nu_{\mu-1}} = -\sqrt{\frac{\nu_\mu}{2}}\,(\zeta^n_\mu - \zeta^{n'}_\mu).$$

Now let us consider the excited states n with only one nucleon on the first excited level, and such that there are no quanta of the surface vibrations ($\nu = 0$). Then in the summation (19) over $n'\nu'$ there will remain only the terms with $F^{n_0\nu}_{n0}$. Moerover, considering Eq. (22b) and retaining in Eq. (19) only terms of order not higher than $(\zeta^n_\mu - \zeta^{n_0}_\mu)^2$, we get $F^{n_0\nu'}_{n,0} = \hbar/2 \sum_\mu \omega^{n_0}_\mu L_{nn_0}(\mu) \times (\zeta^n_\mu - \zeta^{n_0}_\mu)^2$ and the summation (19) will be expressed as follows:

$$1 = \frac{1}{2}\sum_\nu \frac{\left|\sum_\mu \omega^{n_0}_\mu L_{nn_0}(\mu)(\zeta^n_\mu - \zeta^{n_0}_\mu)\right|^2}{(\omega_{n_0\nu} - \omega_{n,0})^2 + {}^1/_4\Gamma^2}. \tag{23}$$

In performing the summation over ν, one should keep in mind that ν_μ can have the values 0, 1; further,

$$\omega_{n_0\nu} - \omega_{n,0} = \Omega_{nn_0} - \sum_\mu \omega^{n_0}_\mu \nu_\mu, \tag{23a}$$

where

$$\Omega_{nn_0} = \frac{E_n - E_{n_0}}{\hbar} + \sum_\mu \Big\{\omega^{n_0}_\mu (\zeta^{n_0}_\mu)^2 - \omega^n_\mu (\zeta^n_\mu)^2 + \frac{1}{2}(\omega^n_\mu - \omega^{n_0}_\mu)\Big\}. \tag{23b}$$

The first term in Eq. (23b) corresponds to the change of energy due to the transition of nucleons from the state n into n_0, under the assumption of a spherical shape for the nucleus. The terms in square brackets account for the change of energy connected with the deformation of a nuclear surface, and the last term accounts for the change of energy of the zeroth vibrations, which happens because of changes in frequencies of vibrations.

Considering the sharp maximum of Eq. (23) at $\Omega_{nn_0} - \sum_\mu \omega^{n_0}_\mu \nu_\mu = 0$, we can write Eq. (23) as follows:

$$1 = \frac{1}{2}\left|\overline{\sum_{\mu''}}\omega^{n_0}_{\mu''} L_{nn_0}(\mu)(\zeta^n_{\mu''} - \zeta^{n_0}_{\mu''})\right|^2 \times \sum_\nu \frac{1}{\left(\Omega_{nn_0} - \sum_\mu \omega^{n_0}_\mu \nu_\mu\right)^2 + {}^1/_4\Gamma^2},$$

where $\overline{\sum_{\mu''}}$ means the average value of the sum over those quantum numbers μ'' for which $\nu_{\mu''} = 1$ and $\sum_{\mu''}\omega^{n_0}_{\mu''}\nu_{\mu''} = \Omega_{nn_0}$. Let $\sum_\mu \omega^{n_0}_\mu \nu_\mu \equiv \Omega$; then,

$$\sum_\nu \frac{1}{\left(\Omega_{nn_0} - \sum_\mu \omega^{n_0}_\mu \nu_\mu\right)^2 + {}^1/_4\Gamma^2} \approx \int \frac{\rho(\Omega)\,d\Omega}{(\Omega_{nn_0} - \Omega)^2 + {}^1/_4\Gamma^2} = \frac{2\pi}{\Gamma}\rho(\Omega_{nn_0}),$$

where $\rho(\Omega)$ is the number of states of surface vibrations per unit of frequency interval. According to Bethe[2]

$$\rho(\Omega) \approx {}^4/_3 R_0^2 \omega^{1/3}(\rho/\gamma)^{2/3}.$$

Hence

$$\Gamma = \pi\rho(\Omega)\left|\overline{\sum_{\mu''}}\omega^{n_0}_{\mu''} L_{nn_0}(\mu)(\zeta^n_{\mu''} - \zeta^{n_0}_{\mu''})\right|^2, \tag{24}$$

where $L_{nn_0}(\mu)$ is determined in Eq. (21a).

For magnitude evaluation of $L_{nn_0}(\mu)$ the change of the function due to deformation of a nuclear surface has to be calculated. Assuming that the potential field for the nucleons in a nucleus can be pictured by a spherical rectangular well of radius R_0 and depth D, the operator of excitation at small deformation of nuclear surface can be represented by

$$W = -DR_0 \sum_p \delta(r_p - R_0) \times \sum_{\lambda\mu}\sqrt{\hbar / B_\lambda \omega_\lambda}\,\xi_{\lambda\mu}(\mathbf{Y}_{\lambda\mu} + \mathbf{Y}^*_{\lambda\mu}).$$

The summation $\sum_p$ extends over all nucleons of a nucleus.

In first approximation of the theory of excitations

$$\varphi_{n_0}(R) = \varphi_{n_0}(R_0) + \sum_m{}' (W_{mn_0}/\hbar\Omega_{n_0m})\,\varphi_m(R_0), \tag{25}$$

where

$$\Omega_{n_0m} = (E_{n_0} - E_m)/\hbar,$$

$$W_{mn_0} = \int \varphi^*_m W \varphi_{n_0}\,d\tau.$$

Substitution of Eq. (25) into Eq. (21a) gives

$$L_{nn_0} = -\sum_{p,\lambda,\mu}\frac{DR_0^3 \Re_n(R_0)\Re_{n_0}(R_0)}{\hbar\Omega_{n_0n}} \times \sqrt{\frac{\hbar}{B_\lambda\omega_\lambda}}\int \mathbf{Y}^*_{n_0}(\mathbf{Y}_{\lambda\mu} + \mathbf{Y}^*_{\lambda\mu})\,\mathbf{Y}_n\,d\Omega.$$

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[2] H. Bethe, Rev. Mod. Phys. 3, 362 (1948)

For simplicity we will consider the nuclei containing only one nucleon outside of the complete shells*. Then the lowest excited states of the nucleus are determined according to the shell model by the transition of this nucleon (in the future we will call it an external nucleon) into some other quantum states. Consequently, the quantum states n_0 and n will differ only by the state of the external nucleon. As it is shown in references 3, the summation

$$\sum_{p} \int \mathrm{Y}^{*}_{lm} \mathrm{Y}_{\lambda\mu} \mathrm{Y}_{lm}\, d\Omega = 0,$$

when it includes all nucleons contained in complete shells. Therefore, only external nucleons will contribute to $L_{nn_0}(\mu)$

$$L_{nn_0}(\mu) = -\frac{DR_0^3 \Re_n(R_0)\, \Re_{n_0}(R_0)}{\hbar \Omega_{n_0 n}} \tag{26}$$

$$\times \sum_{\lambda\mu} \sqrt{\frac{\hbar}{B_\lambda \omega_\lambda}} \int \mathrm{Y}^{*}_{l_0 m_0} (\mathrm{Y}_{\lambda\mu} + \mathrm{Y}^{*}_{\lambda\mu})\, \mathrm{Y}_{\lambda\mu}\, d\Omega.$$

The factor $DR_0^3 \Re_n(R_0) \Re_{n_0}(R_0)$ can be computed under certain assumptions on the potential well in which the nucleons are moving. These computations are very approximate. They were done in works mentioned in reference 3.

It was shown that this factor does not depend much on the state of a nucleon and is approximately equal to 40 mev (at $D = 28.3$ mev).

Let g denote the integral $\int \mathrm{Y}^{*}_{l_0 m_0} \mathrm{Y}_{\lambda\mu} \mathrm{Y}_{lm}\, d\Omega$. This integral is zero when the selection rules $|\lambda - l_0| \leq l \leq |\lambda + l_0|$ and $\lambda + l + l_0 = 2n$ (where n is an integer) do not hold. For the transition $f \rightleftarrows p$ the average value of second power of g^2 according to the computation of Reifman[4] is $g^2 \leq 0.1$.

Hence, the following approximation can be made:

$$|L_{n_0 n}|^2 \approx \frac{0.4\, \hbar}{B\omega} \left(\frac{40}{E_{\mathrm{mev}}}\right)^2,$$

where E_{mev} is the energy of the single nucleonic transition expressed in mev and ω is the frequency of surface vibrations.

By substitution of these results in Eq. (24) we get

$$\Gamma = \frac{0.4\, \pi\rho(\Omega)\, \hbar}{B} \left(\frac{40}{E_{\mathrm{MeV}}}\right)^2 \left| \sum_{\mu} \omega_{\mu}^{1/2} (\zeta_{\mu}^{n} - \zeta_{\mu}^{n_0}) \right|^2.$$

Thus, the probability of a nonradiative transfer of energy of a single nucleonic excitation to the collective degrees of freedom of a nucleus is the greater, the greater is the square of the difference $(\zeta_\mu^n - \zeta_\mu^{n_0})$ characterizing the change of equilibrium configuration of a nucleus during the transition of a nucleon from an excited into the ground state. Such nonradiative transitions must widen the levels which correspond to the excitation of separate nucleons in a nucleus. The total width will be equal to the sum of widths of the nonradiative transition processes, considered above, γ-emission, conversion, etc.

For the evaluation of magnitude of Γ we assume that the transition takes place from a spherically symmetrical state n_0 (or into spherically symmetrical state), and that the interaction exists only with the surface vibrations $\lambda = 2$; then,

$$\Gamma \approx 0.4\, \pi\rho(\Omega) \left(\frac{40}{E_{\mathrm{MeV}}}\right)^2 \omega^2 \beta,$$

where ω is the frequency of vibrations of the nuclear surface at $\lambda = 2$. The value

$$\beta = \sum_{\mu} \frac{\hbar}{B\omega} (\zeta_\mu^n)^2 = \sum_{\mu} |\alpha_\mu|^2$$

determines the ellipsoidal deformation of a nuclear surface. According to reference 1, it can be calculated from the value of a quadruple moment of a nucleus Q_0 (with respect to the axis of symmetry of a nucleus) by using the formula $\beta = (5\pi)^{1/2} Q_0 / 3ZR_0^2$ or from the moment of inertia of a nucleus $J = 3B\beta^2$. Further, from Eq. (24a) it follows that $\rho(\Omega) \approx 2.6 \times 10^{-30} \omega^{1/3} A^{2/3}$. For example, by letting $E = 1.5$ mev, $\omega = 8 \times 10^{19}$; and $A = 216$; $\beta = 0.1$, we will get $\Gamma \approx 2 \times 10^{20}$ which corresponds to the width of a level ≈ 0.1 mev. Knowing the magnitude of Γ, one can determine the mean free path of a nucleon in a nucleus $\Lambda = v/\Gamma$, where v is the velocity of a nucleon. Since Γ differs from zero only for the excited states of a nucleus, $\Lambda \neq \infty$ only for the excited states of a nucleus.

In some works[5] it was shown that the experimental data[6] about the scattering of neutrons with the energy less than 3 mev can be phenomenologically explained as a scattering in a complex potential, the imaginary part of which is responsible for the formation of an excited compound nucleus. This verifies the notion that during the first stages of nuclear reactions the average field of a nucleus acts on the incident nucleon. In this case the mean free path, which is connected with the probability of formation of an intermediate nucleus, can be determined in an analogous way.

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* The results obtained below apply only to nuclei which lack one nucleus to fill the outer shell.

[3] J. Rainwater, Phys. Rev. **79**, 412 (1950); E. Feenberg and K. Hammak, Phys. Rev. **81**, 285 (1951); R. Wageningen and J. Boer, Physica **18**, 369 (1952)

[4] A. Reifman, Z. Naturforsch **8a**, 505 (1953)

[5] V. Weisskopf, Physica **18**, 1083 (1952); H. Feshbach, C. Porter and V. Weisskopf, Phys. Rev. **90**, 166 (1953)

[6] H. Barschall, Phys. Rev. **86**, 431 (1952)

Of course, one should keep in mind that the possibility of a separation of single nucleonic excitations is determined by conditions, which make the inequality $\hbar\Gamma < \Delta$ valid. Here Δ is the distance between the neighboring single particle levels. If this inequality is violated, which apparently must be the case at high excitation energies of a nucleus, the separation of single particle excitations from a total excitation of a nucleus is entirely impossible.

4. EFFECT OF THE DEFORMATION OF A NUCLEAR SURFACE ON THE PROBABILITY OF SINGLE NUCLEONIC EXCITATIONS

In the adiabatic approximation the state of a nucleus is characterized by the energy levels $E_{n\nu}$ (16) and wave functions

$$\psi_{n\nu} = \varphi_n(r, R)\prod_\mu \psi_{\nu_\mu}(\xi_\mu - \zeta_\mu^n).$$

We assume that under the influence of an excitation, the operator $A(r)$ of which operates only on the coordinates of the nucleons, the transition from the state $n_0\nu$ into the state $n\nu$ takes place without change of the quantum state $\nu = (\ldots 0_\mu \ldots)$ of the vibration of a nuclear surface. The probability of the transition within one second will be equal to

$$W_{nn_0} = \frac{2\pi}{\hbar}\left|\int \varphi_n^*(r)\,A(r)\,\varphi_{n_0}(r)\,dr\right|^2 \times\left\{1 - {}^1\!/_4\sum_\mu(\zeta_\mu^n - \zeta_\mu^{n_0})^2\right\}.$$

The ratio of the probability of a transition $n_0 \to n$, upon consideration of possible deformation of a nucleus, to the probability of the transition computed for the case of a rigid nuclear shell will be equal to

$$P = 1 - {}^1\!/_4\sum_\mu(\zeta_\mu^n - \zeta_\mu^{n_0})^2;$$

The summation is performed over all possible types of vibrations of a nuclear surface. The last expression can be simplified when one of the states, between which the transition takes place, corresponds to the spherically symmetrical nucleus; then $P = 1 - 1/4\sum_\mu \zeta_\mu^2$. By assuming, for simplicity, that the greatest interaction is accomplished with the surface vibrations $\lambda = 2$, we get $P = 1 - (B\omega\beta/4\hbar)$. When $A = 216$, $\Omega = 8 \times 10^{19}$, $\beta = 0.3$, we get $P \approx 0.8$.

Translated by G. Filipovich
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The Gauge Transformation of the Green's Function for Charged Particles

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A gauge transformation is carried out on the Green's function and the vertex operator for charged particles interacting with an electromagnetic field.

GAUGE invariance arises in the field theory of charged particles interacting with an electromagnetic field. Given a gauge transformation of the potential of the electromagnetic field

$$A_\mu \to A_\mu + \partial \varphi / \partial x_\mu \tag{1}$$

(ϕ is an arbitrary operator function), the ψ-function of the particle is transformed as follows:

$$\psi \to \psi e^{ie\varphi} \tag{2}$$

(e is the charge, $\hbar$ and c are taken to be unity).

We shall attempt to determine here how the Green's function for the particles will change under such a gauge transformation. The Green's function for particles is well-known to be

$$G(xx') = \langle (\psi(x) \overline{\psi}(x'))_+ \rangle. \tag{3}$$

The brackets denote a vacuum expectation value.

The Green's function $G(xx')$ will change under the gauge transformation (1). From Eq. (2), it can be written in the form:

$$G(xx') = G_0(xx') \langle (e^{ie\varphi(x)} e^{-ie\varphi(x')})_+ \rangle, \tag{4}$$

where $G_0(xx')$ stands for the Green's function for the particular case when the longitudinal (in the four-dimensional sense) part of the photon's Green's function is equal to zero[1]. The Fourier components of the Green's function for photons

$$D_{\mu\nu}(xx') = i \langle (A_\mu(x) A_\nu(x'))_+ \rangle \tag{5}$$

can be written in the general case in the form[1]

$$D_{\mu\nu}(k) = 4\pi d_t \frac{1}{k^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + 4\pi d_l \frac{k_\mu k_\nu}{k^4}. \tag{6}$$

The terms containing d_t and d_l represent respectively the transverse and longitudinal parts of the function $D_{\mu\nu}$. Now let us compute the vacuum expectation value of the chronological product. The longitudinal part of Green's function turns out to be the original function. It does not depend upon interaction with the field. Thereby the problem reduces to calculating the vacuum expectation value of the expression $e^{ie\phi(x)} e^{-ie\phi(x')}$, where the operators ϕ represent a free field. Expanding the free field ϕ into plane waves, we have

$$\varphi = \sum_k \varphi_k = \sum \lambda(k^2) (a_k e^{ikx} + a_k^+ e^{-ikx}). \tag{7}$$

a_k and a_k^+ represent respectively creation and annihilation operators for longitudinal photons; $\lambda(k^2)$, the amplitude which characterizes the contribution of the photons, by the four-dimensional wave vector k. We also include in $\lambda(k^2)$ a normalization factor. In view of the smallness of this factor (it contains the reciprocal volume) we shall, in what follows, leave out terms of higher power than λ^2. Expressions of the type $e^{ie\phi}$ can be expanded by means of Eq. (7) into the infinite product

$$e^{ie\varphi(x)} = \prod_k e^{ie\varphi_k}, \quad e^{-ie\varphi(x')} = \prod_l e^{-ie\varphi'_l}. \tag{7'}$$

The operators a_k and a_k^+ corresponding to different values of the wave vectors commute with each other. Let us now take advantage of this circumstance and take into account the fact that the vacuum expectation values of terms of first order in the operator a_k (or ϕ_k) are zero. The vacuum expectation values of the products $\phi_k \phi'_l$, in accordance with what we said above, can differ from zero only if $k = l$. With the help of Eq. (7′) we obtain for the vacuum expectation value of the expression $e^{ie\phi(x)} e^{-ie\phi(x')}$:

$$\langle (e^{ie\varphi(x)} e^{-ie\varphi(x')})_+ \rangle = \left\langle \left(\prod_k \left(1 + ie\varphi_k - \frac{e^2}{2} \varphi_k^2 \right) \prod_l \left(1 - ie\varphi'_l - \frac{e^2}{2} \varphi'^2_l \right) \right)_+ \right\rangle \tag{8}$$

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[1] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 95, 773 (1954)

$$= \prod_k (1 - \frac{e^2}{2}\langle\varphi_k^2\rangle] - \frac{e^2}{2}\langle\varphi_k'^2\rangle + e^2\langle(\varphi_k\varphi_k')\rangle$$

$$= \exp\left\{-\frac{1}{2}e^2\sum_k \underline{\langle\varphi_k^2\rangle} + \langle\varphi_k'^2\rangle - 2\langle(\varphi_k\varphi_k')_+\rangle\right\}.$$

As we have already noted, the vacuum expectation value of the product of the two operators a_k and a_k^+ is always zero except when $k = l$. Thereby the Green's function for the ϕ field becomes

$$\Delta_F(xx') = i\langle(\varphi(x)\varphi(x'))_+\rangle = i\langle(\varphi_k\varphi_k')_+\rangle. \quad (9)$$

Considering Eq. (9), we rewrite Eq. (8) in the form

$$\langle(e^{ie\varphi(x)}e^{-ie\varphi(x')})_+\rangle$$
$$= \exp\{ie^2(\Delta_F(0) - \Delta_F(xx')\}. \quad (10)$$

Now let us apply the results we have obtained and write the final formula for the Green's function $G(xx')$ for charged particles; from Eqs. (4) and (10) we have

$$G(xx') = G_0(xx')$$
$$\times \exp\{ie^2(\Delta_F(0) - \Delta_F(xx')\}. \quad (11)$$

Formula (11) links the Green's function for charged particles $G(xx')$ with its value $G_0(xx')$ computed under the assumption that the longitudinal part of the photon D-function is equal to zero. The longitudinal part of the Green's function for photons is connected with the Δ-function of the ϕ field, as evidenced by the expression:

$$D_{F\mu\nu}(xx') = \frac{\partial^2}{\partial x_\mu \partial x_\nu'}\Delta_F(xx'). \quad (12)$$

In accordance with Eq. (6), the expression for the Fourier-components becomes

$$\Delta_F(k) = 4\pi\frac{d_l(k)}{k^4}. \quad (13)$$

It is not generally possible to write in Fourier components form the Green's function as it appears in formula (11). It is possible, however, to obtain a formula for the change in the Fourier components of the Green's function for particles, for an infinitesimal gauge transformation of the potential. Performing a variation on Eq. (11) we obtain

$$\delta G(xx') = ie^2 G(xx')(\delta\Delta_F(0)$$
$$- \delta\Delta_F(xx')). \quad (14)$$

We rewrite this expression in Fourier components; with the help of Eq. (13), we find

$$\delta G(p) = \frac{ie^2}{\pi}\int\frac{\delta d_l(k)}{k^4}\{G(p) - G(p-k)\}\,d^4k. \quad (15)$$

Let us apply the resulting formula to the case of spin ½ particles. The unperturbed Green's function of the free particle is given by

$$G_0(p) = 1/(\hat{p} - m), \quad \hat{p} = \gamma_\mu p_\mu.$$

Let $d_l(k)$ be a slowly varying function of the argument (k^2), so that the condition $\frac{e^2}{\pi} d_l(kl \gg 1$ is satisfied. Let us substitute into the right side of Eq. (15) the unperturbed value of the function $G(p)$. For slow variations of the function $d_l(k)$ in Eq. (15), significant contributions to the integral come from the region $k^2 \gg p^2$. Because of this, the term $G(p\text{-}k)$ may be neglected with respect to $G(p)$ on the right side of Eq. (15). Equation (15) is then satisfied by the Green's function

$$G(p) = \frac{\beta(p^2)}{(\hat{p} - m)} \quad (16)$$

where $\beta(p^2)$ is a slowly varying function of its argument.

For the case when $p^2 \gg m^2$, from Eq. (15) and from the expression

$$d^4k = \frac{i}{4}(-k^2)\,d(-k^2),$$

we obtain

$$\delta\beta(p^2) = -\frac{e^2}{4\pi}\beta(p^2)\int_{-p^2}^{\infty}\frac{\delta d_l}{-k^2}d(-k^2). \quad (17)$$

Let us denote $\beta(p^2)$ when $d_l = 0$ by the expression $\beta_t(p^2)$. Calculations show that in the case of spin ½, we have $\beta_t(p^2) = 1$ [1].

From Eq. (17) we obtain for a finite gauge transformation the well-known formula

$$\beta(p^2) = \exp\left\{-\frac{e^2}{4\pi}\int_{-p^2}^{\infty} d_l\frac{d(-k^2)}{-k^2}\right\}. \quad (18)$$

The problem for spin zero can be solved in an analogous fashion. In this case the Green's function is written in the form

$$G(p) = \frac{\beta(p^2)}{p^2 - m^2}. \quad (19)$$

Calculations show that for slowly varying d_l, the quantity $\beta(p^2)$ similarly turns out to be a slowly varying function. For a finite gauge transformation we obtain, analogously to Eq. (18),

$$\beta(p^2) = \beta_t(p^2) \exp\left\{ -\frac{e^2}{4\pi} \int\limits_{-p^2}^{\infty} d_l \frac{d(-k^2)}{-k^2} \right\}, \tag{20}$$

where $\beta_t(p^2)$ denotes $\beta(p^2)$ when $d_l = 0$.

Let us now turn to the gauge transformation of the vertex operator $\Gamma_\mu(xx'; \xi)$. We shall start from the vacuum expectation value of the chronological product

$$\langle (\psi(x) A_\mu(\xi) \overline{\psi}(x'))_+ \rangle. \tag{21}$$

The vertex operator $\Gamma_\mu(xx'; \xi)$ is linked to the function $B_\mu(xx'; \xi)$ by the integral

$$B_\mu(xx'; \xi) = e^2 \int G(xx'') \Gamma_\nu(x''x'''; \xi') G(x'''x') \times D_{\nu\mu}(\xi'\xi)\, d^4x''\, d^4x'''\, d^4\xi'. \tag{22}$$

Under gauge transformation, the function $B_\mu(xx'; \xi)$ changes as follows:

$$B_\mu(xx'; \xi) \to B_{0\mu}(xx'; \xi) \langle (e^{ie\varphi(x)}\, e^{-ie\varphi(x')})_+ \rangle + G_0(xx') \left\langle \left(e^{ie\varphi(x)} \frac{\partial\varphi(\xi)}{\partial\xi_\mu} e^{-ie\varphi(x')} \right)_+ \right\rangle. \tag{23}$$

Here $B_{0\mu}(xx'; \xi)$ denotes the function B_μ when the longitudinal part of the photon D-function is equal to zero. The vacuum expectation value of the product which appears in the term containing $B_{0\mu}$ in Eq. (23) is calculated from Eq. (10). As for the factor which appears with $G_0(xx')$, simple calculations involving Eqs. (7) and (8) yield for it

$$\left\langle \left(e^{ie\varphi(x)} \frac{\partial\varphi(\xi)}{\partial\xi_\mu} e^{-ie\varphi(x)} \right)_+ \right\rangle = \exp\{ie^2 [\Delta_F(0) - \Delta_F(xx')]\} \times e \frac{\partial}{\partial\xi_\mu} (\Delta_F(x\xi) - \Delta_F(\xi x')). \tag{24}$$

Substituting Eq. (24) in Eq. (23) and taking Eq. (11) into account, we finally obtain

$$B_\mu(xx'; \xi) = B_{0\mu}(xx'; \xi) \exp\{ie^2(\Delta_F(0) - \Delta_F(xx'))\} + B_{1\mu}(xx'; \xi), \tag{25}$$

$$B_{1\mu}(xx'; \xi) = eG(xx') \frac{\partial}{\partial\xi_\mu} (\Delta_F(x\xi) - \Delta_F(\xi x')). \tag{26}$$

Let us clarify the connection between each of the terms in Eq. (25) and the Green's functions for particles and photons. First we show that the second term in Eq. (25) coincides exactly with that part of Eq. (22) which corresponds to the longitudinal part of the photon function $D^l_{\mu\nu}$:

$$B_{1\mu}(xx'; \xi) = e \int G(xx'') \Gamma_\nu(x''x'''; \xi') \times G(x'''x') D^l_{\nu\mu}(\xi'\xi)\, d^4x''\, d^4x'''\, d^4\xi'. \tag{27}$$

This expression can be obtained rigorously by going to Fourier components. Using Eq. (26), the Fourier components of $B_{1\mu}(xx'; \xi)$ become

$$B_{1\mu}(p, p-k; k) = e(G(p) - G(p-k)) k_\mu \Delta_F(k). \tag{28}$$

The Fourier component of the right side of Eq. (27) is evidently given by [see Eq. (6)]:

$$eG(p)\Gamma_\nu(p, p-k; k) G(p-k) \frac{k_\nu k_\mu}{k^4} 4\pi d_l(k) \tag{29}$$

We shall now make use of the famous generalized theorem of Ward[2]

$$k_\nu \Gamma_\nu(p, p-k; k) = -(G^{-1}(p) - G^{-1}(p-k)). \tag{30}$$

Expression (29) then becomes:

$$e(G(p) - G(p-k)) \frac{k_\mu}{k^4} 4\pi d_l(k). \tag{31}$$

Comparison of Eqs. (31) and (28), in view of Eq. (13), verifies the correctness of Eq. (27). If Eq. (27) is proved, then it follows from Eqs. (22) and (25) that

$$B_{0\mu}(xx'; \xi) \exp\{ie^2 [\Delta_F(0) - \Delta_F(xx')]\} = e \int G(xx'') \Gamma_\nu(x''x'''; \xi') G(x'''x') \times D^t_{\nu\mu}(\xi'\xi)\, d^4x''\, d^4x'''\, d^4\xi' \tag{32}$$

($D^t_{\mu\nu}$ is the transverse part of the D-function).

Rewriting this expression for an infinitesimal gauge transformation and going to Fourier components, we find

$$ie^2 \int \{G(p)\Gamma_\mu(p, p-k; k) G(p-k) - G(p-r)\Gamma_\mu(p-r, p-r-k; k) \times G(p-k-r)\} \frac{\delta d_l(r)}{r^4} d^4r = G(p)\delta\Gamma_\mu(p, p-k; k) G(p-k) \tag{33}$$

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[2] H. Green, Proc. Phys. Soc. 66, 837 (1953)

$$+ie^2\int\{[G(p)-G(p-r)]$$

$$\times\Gamma_\mu(p,p-k;k)\,G(p-k)$$

$$+G(p)\,\Gamma_\mu(p,p-k;k)\,[G(p-k)$$

$$-G(p-k-r)]\}\,\frac{\delta d_l(r)}{r^4}\,d^4r.$$

Solving for $\delta\Gamma_\mu$ we finally obtain

$$G(p)\,\delta\Gamma_\mu(p,p-k;k)\,G(p-k)$$

$$=-ie^2\int\{G(p)\,\Gamma_\mu(p,p-k;k)(G(p-k) \tag{34}$$

$$-G(p-k-r))+G(p-r)$$

$$\times[\Gamma_\mu(p-r,p-r-k;k)\,G(p-k-r)$$

$$-\Gamma_\mu(p,p-k;k)\,G(p-k)]\}\,\frac{\delta d_l(r)}{r^4}\,d^4r.$$

For the case of spin ½ particles, the change in the vertex operator $\Gamma_\mu(p,\ p-k;\ k)$ under infinitesimal gauge transformation ($p^2 \gg m^2$) can be found by a method similar to the one that was used for the Green's function. If $d_l(r)$ is a slowly varying function in Eq. (34), then all the terms on the right side except the first can be neglected (upon integration in the significant region of large r^2). After this it is easily found (assuming that $(p-k)^2$, p^2 and k^2 are all of the same order of magnitude) that

$$\delta\Gamma_\mu(p,p-k;k) \tag{35}$$

$$=-ie^2\,\Gamma_\mu(p,p-k;k)\int_{-p^2}^{\infty}\frac{\delta d_l(r)}{-r^2}\,d(-r^2).$$

This result is found to be in conformity with Ward's theorem.

In conclusion we wish to extend our thanks to A. A. Abrikosov for taking part in discussions with us.

Translated by M. A. Melkanoff
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The Theory of Molecular Attractive Forces between Solids

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A macroscopic theory is developed for the interaction of bodies whose surfaces are brought within a small distance of one another. The interaction is considered to come about through the medium of the fluctuating electromagnetic field. The limiting cases of separations small and large compared with the wavelengths of the absorption bands of the solid are studied. Upon going to the limiting case of rarefied media, the van der Waals forces of interaction between individual atoms are obtained. The effect of temperature on the interaction of the bodies is considered.

IT is well-known that the forces of interaction between neutral atoms, located at a distance R from one another which is large compared to their internal dimensions, leads to an attraction inversely proportional to R^7. These so-called van der Waals forces are obtained in the second approximation of perturbation theory, applied to the electrostatic interaction of two dipoles. Such a treatment is however valid only so long as the separation R is small compared to the wavelengths λ corresponding to transitions between the ground and excited states of the atom. For $R \gtrsim \lambda$, retardation effects become important. The interaction of atoms when these effects are taken into account was studied by Casimir and Polder [1]. Here the perturbation operator is the sum of the electrostatic interaction of the atoms and their interactions with the radiation field. The latter can, in the usual fashion, be regarded as the result of emission and absorption of virtual quanta. With respect to this interaction, the perturbation theory must be applied up to fourth order terms, and the calculations become rather unwieldy. In the limitting case of $R \gg \lambda$, the attractive force turns out to be proportional to R^{-8} rather than to R^{-7}.

The presence of attractive forces between neutral atoms naturally results in the presence of similar forces between two macroscopic bodies whose surfaces are brought to within a small distance of one another. However, the calculation of these forces, starting from the known interaction of the individual atoms, would be possible only for sufficiently rarefied bodies, i. e., for gases – a case which of course cannot be realized practically. We can however approach this problem in purely macroscopic fashion (since the distance between the bodies is assumed to be large compared to interatomic distances). From this point of view, the interaction of the objects is regarded as occurring through the medium of the fluctuating electromagnetic field which is always present in the interior of any absorbing medium, and also extends beyond its boundaries, – partially in the form of travelling waves radiated by the body, partially in the form of standing waves which are damped exponentially as we move away from the surface of the body. It must be emphasized that this field does not vanish even at absolute zero, at which point it is associated with the zero point vibrations of the radiation field.

The method for calculating interaction forces which is based on these considerations has full generality, and is applicable to any body at any temperature. It also automatically takes into account retardation effects, which become important for sufficiently large separations between the bodies. In the limiting case of rarefied media, the method must of course lead to the same results as are obtained by considering the interactions of individual atoms.

1. CALCULATION OF THE FLUCTUATING ELECTROMAGNETIC FIELD

We picture the interacting bodies as two media filling half spaces with plane-parallel boundaries separated from one another by a distance l (Fig. 1). To calculate the fluctuating field in the interior of the two media, we shall use the general theory which is due to Rytov and is described in detail in his book[2]. This method is based on the introduction into the Maxwell equations of a "random" field (just as, for example, one introduces a "random" force in the theory of Brownian motion).

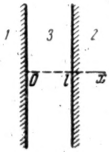

FIG. 1

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[1] H. B. G. Casimir and D. Polder, Phys. Rev. 73, 360 (1948)

[2] S. M. Rytov, *Theory of Electrical Fluctuations and Thermal Radiation*, Publishing House, Academy of Sciences, USSR, 1953

In a dielectric, nonmagnetic medium, these equations are, for a monochromatic field (time factor $e^{-i\omega t}$)*

$$\operatorname{curl} \mathbf{E} = i\frac{\omega}{c}\mathbf{H}, \tag{1.1}$$

$$\operatorname{curl} \mathbf{H} = -i\frac{\omega}{c}\varepsilon\mathbf{E} - i\frac{\omega}{c}\mathbf{K},$$

where $\epsilon = \epsilon(\omega)$ is the complex dielectric constant, and $\mathbf{K}$ is the random field. The fundamental characteristic of the latter is the correlation function, determining the average value of the product of components of $\mathbf{K}$ at two different points in space. By the very nature of the introduction of a random field in a macroscopic fluctuation theory, in which atomic distances are considered to be negligibly small, this correlation has the character of a δ-function. According to Rytov, it is given by the formula

$$\overline{K_i(x, y, z) K_k(x', y', z')} \tag{1.2}$$

$$= A\varepsilon''(\omega)\delta_{ik}\delta(x - x')\delta(y - y')\delta(z - z'),$$

$$A = 4\hbar\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/T} - 1}\right) = 2\hbar\coth\frac{\hbar\omega}{2T},$$

where T is the temperature and ϵ'' is the imaginary part of $\epsilon = \epsilon' + i\epsilon''$ (for the quasistationary range of frequencies, an analogous formula was obtained by Leontovich and Rytov[4]).

We represent the function $\mathbf{K}(x, y, z)$ in the form of a Fourier integral, which we write for the half space $x < 0$ in the form:

$$\mathbf{K}(x, y, z) = \int_{-\infty}^{+\infty} \mathbf{g}(\mathbf{k}) e^{i\mathbf{q}\cdot\mathbf{r}} \cos k_x x \, d\mathbf{k}. \tag{1.3}$$

Here and in the sequel we denote by $\mathbf{q}$ a two-dimensional vector with components k_y, k_z (so that $k^2 = k_x^2 + q^2$); and by $\mathbf{r}$, the radius vector in the y - z plane. For the Fourier components $\mathbf{g}(\mathbf{k})$, the correlation function corresponding to the spatial correlation (1.2) is (cf. reference 2, No. 4):

$$\overline{g_i(\mathbf{k}) g_k(\mathbf{k}')} = \frac{A\varepsilon''}{4\pi^3}\delta_{ik}\delta(\mathbf{k} - \mathbf{k}'). \tag{1.4}$$

We now proceed to the solution of Eq. (1.1) with the appropriate boundary conditions on the surfaces of the two bodies. In medium 1 ($x < 0$), we look for fields $\mathbf{E}$ and $\mathbf{H}$ of the form:

$$\mathbf{E}_1 = \int_{-\infty}^{+\infty} \{\mathbf{a}_1(\mathbf{k}) \cos k_x x + i\mathbf{b}_1(\mathbf{k}) \sin k_x x\} e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{k}$$

$$+ \int_{-\infty}^{+\infty} \mathbf{u}_1(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r} - is_1 x} d\mathbf{q},$$

$$\mathbf{H}_1 = \frac{c}{\omega}\int_{-\infty}^{+\infty} \{([\mathbf{q}\mathbf{a}_1] + k_x[\mathbf{n}\mathbf{b}_1]) \cos k_x x \tag{1.5}$$

$$+ i([\mathbf{q}\mathbf{b}_1] + k_x[\mathbf{n}\mathbf{a}_1]) \sin k_x x\} e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{k}$$

$$+ \frac{c}{\omega}\int_{-\infty}^{+\infty} \{[\mathbf{q}\mathbf{u}_1] - s_1[\mathbf{n}\mathbf{u}_1]\} e^{i\mathbf{q}\cdot\mathbf{r} - is_1 x} d\mathbf{q},$$

where $\mathbf{n}$ is a unit vector in the direction of the x axis, and

$$s_1 = \sqrt{\frac{\omega^2}{c^2}\varepsilon_1 - q^2}, \tag{1.6}$$

where the sign of the root is to be chosen so that the imaginary part of s will be positive**. We have here made use of the first of Eqs. (1.1).

The first terms in these expressions represent a solution of the inhomogeneous Eqs. (1.1). Substituting them in the second equation of (1.1) and writing $\mathbf{K}$ in the form (1.3), we find the following relations, expressing $\mathbf{a}_1$ and $\mathbf{b}_1$ in terms of the Fourier components $\mathbf{g}_1$ of the random field:

$$\mathbf{a}_1 = \frac{1}{\varepsilon_1(k^2 - \omega^2\varepsilon_1/c^2)}\Big[\frac{\omega^2}{c^2}\varepsilon_1\mathbf{g}_1$$

$$- \mathbf{q}(\mathbf{q}\cdot\mathbf{g}_{1r}) - k_x^2 g_{1x}\mathbf{n}\Big], \tag{1.7}$$

$$\mathbf{b}_1 = -\frac{k_x}{\varepsilon_1(k^2 - \omega^2\varepsilon_1/c^2)}[\mathbf{n}(\mathbf{q}\cdot\mathbf{g}_{1r}) + \mathbf{q}g_{1x}].$$

Two-dimensional vectors in the y-z plane ar indicated by the subscript r from now on.

The second integrals in (1.6) represent the solution of the homogeneous equations (1.1) (i.e., the equations with $\mathbf{K}$ omitted), and describe the plane wave field reflected from the boundary of the medium. The condition for transversality of these waves is.

$$\mathbf{u}_{1r}\cdot\mathbf{q} - s_1 u_{1x} = 0. \tag{1.8}$$

In the second medium (the half space $x > l$), the field $\mathbf{E}_2$, $\mathbf{H}_2$ is given by the same formulas (1.5), (1.7), (1.8), with the index 1 changed to 2,

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* The question of the meaning of monochromatic components for quantities which are not expandable in the usual sense in a Fourier integral, (as is the case for the fluctuation field) is discussed in reference 2, No. 2, and reference 3, No. 117.

[3] L. D. Landau and E. Lifshitz, Statistical Physics, 3rd Edition, Gostekhizdat, 1951.

[4] M. A. Leontovich and S. M. Rytov, J. Exper. Theoret. Phys. USSR, 23, 246 (1952)

** Since the imaginary part of the expression under the square root sign ($\omega^2\epsilon''/c^2$) is positive, when Im $s > 0$ we also have Re $s > 0$).

$\cos k_x x$, $\sin k_x x$ replaced by $\cos k_x(x-l)$, $\sin k_x(x-l)$ and change in the sign of s (the "reflected" waves now propagate along the positive x direction). Finally, in the space between the media (vacuum), we have $\epsilon = 1$, $\mathbf{K} = 0$, and the field is given by the general solution of the homogeneous equations, which we write in the form:

$$\mathbf{E}_3 = \int\limits_{-\infty}^{+\infty} \{\mathbf{v}(\mathbf{q})\, e^{ipx} + \mathbf{w}(\mathbf{q})\, e^{-ipx}\}\, e^{i\mathbf{q}\mathbf{r}} d\mathbf{q}, \tag{1.9}$$

$$\mathbf{H}_3 = \frac{c}{\omega} \int\limits_{-\infty}^{+\infty} \{([\mathbf{q}\mathbf{v}] + p[\mathbf{n}\mathbf{v}])\, e^{ipx}$$

$$+ ([\mathbf{q}\mathbf{w}] - p[\mathbf{n}\mathbf{w}])\, e^{-ipx}\}\, e^{i\mathbf{q}\mathbf{r}} d\mathbf{q},$$

where

$$p = \sqrt{\frac{\omega^2}{c^2} - q^2}, \tag{1.10}$$

and $\mathbf{v}$ and $\mathbf{w}$ satisfy the transversality conditions

$$\mathbf{v}_r \cdot \mathbf{q} + p v_x = 0, \quad \mathbf{w}_r \cdot \mathbf{q} - p w_x = 0. \tag{1.11}$$

The boundary conditions on the surfaces of the media are the requirement of continuity of the tangential components of $\mathbf{E}$ and $\mathbf{H}$. On the plane $x = 0$, this gives the following equations:

$$\int\limits_{-\infty}^{+\infty} \mathbf{a}_{1r}\, dk_x + \mathbf{u}_{1r} = \mathbf{v}_r + \mathbf{w}_r, \tag{1.12}$$

$$\int\limits_{-\infty}^{+\infty} (\mathbf{q}\, a_{1x} - k_x \mathbf{b}_{1r})\, dk_x + \mathbf{q}\, u_{1x} + s_1 \mathbf{u}_{1r}$$

$$= \mathbf{q}(v_x + w_x) - p(\mathbf{v}_r - \mathbf{w}_r).$$

The conditions at the plane $x = l$ differ in having s_1, $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{v}$, $\mathbf{w}$ replaced by s_2, $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{v}e^{ipl}$, $\mathbf{w}e^{-ipl}$, respectively.

The collection of boundary conditions and continuity equations determines all the field amplitudes. In what follows, we shall need only the field between the two media. For a given value of $\mathbf{q}$, we resolve $\mathbf{v}_r$ and $\mathbf{w}_r$ along the mutually perpendicular vectors $\mathbf{q}$ and $\mathbf{n}\mathbf{q}$, which we choose as y and z axes, respectively. The calculation leads to the following formulas for the components of $\mathbf{v}$ and $\mathbf{w}$, expressed in terms of the amplitudes $\mathbf{g}$ of the random field:

$$v_y = \int\limits_{-\infty}^{+\infty} \frac{p}{\Delta} \Big\{ s_1 e^{-ipl} (\varepsilon_2 p + s_2) \frac{q g_{1x} - s_1 g_{1y}}{k_x^2 - s_1^2} \tag{1.13}$$

$$+ s_2(\varepsilon_1 p - s_1) \frac{q g_{2x} + s_2 g_{2y}}{k_x^2 - s_2^2} \Big\} dk_x,$$

$$w_y = \int\limits_{-\infty}^{+\infty} \frac{p}{\Delta} \Big\{ -s_1 e^{ipl} (\varepsilon_2 p - s_2) \frac{q g_{1x} - s_1 g_{1y}}{k_x^2 - s_1^2}$$

$$- s_2(\varepsilon_1 p + s_1) \frac{q g_{2x} + s_2 g_{2y}}{k_x^2 - s_2^2} \Big\} dk_x,$$

$$v_z = \int\limits_{-\infty}^{+\infty} \frac{\omega^2}{c^2 \Delta'} \Big\{ -s_1 e^{-ipl} (s_2 + p) \frac{g_{1z}}{k_x^2 - s_1^2}$$

$$+ s_2(s_1 - p) \frac{g_{2z}}{k_x^2 - s_2^2} \Big\} dk_x,$$

$$w_z = \int\limits_{-\infty}^{+\infty} \frac{\omega^2}{c^2 \Delta'} \Big\{ s_1 e^{-ipl} (s_2 - p) \frac{g_{1z}}{k_x^2 - s_1^2}$$

$$- s_2(s_1 + p) \frac{g_{2z}}{k_x^2 - s_2^2} \Big\} dk_x,$$

$$v_x = -\frac{q v_y}{p}, \qquad w_x = \frac{q w_y}{p},$$

where we have introduced the notation:

$$\Delta = e^{ipl}(s_1 - \varepsilon_1 p)(s_2 - \varepsilon_2 p)$$

$$- e^{-ipl}(s_1 + \varepsilon_1 p)(s_2 + \varepsilon_2 p),$$

$$\Delta' = e^{ipl}(s_1 - p)(s_2 - p)$$

$$- e^{-ipl}(s_1 + p)(s_2 + p).$$

The quantity q runs through values from zero to infinity, while p runs through real values from ω/c to zero, and pure imaginary values from zero to $i\infty$. The first correspond to undamped plane waves in the space between the two media, while the second refer to exponentially damped (so-called "inhomogeneous") plane waves.

2. CALCULATION OF THE FORCE OF ATTRACTION

We shall calculate the force F of mutual attraction, acting on unit surface of each of the bodies, as the xx-component of the Maxwell stress tensor. The tensor calculated from the expressions obtained above for monochromatic field components must still be integrated over all frequencies. For the definition of the time factor which has been used, in particular in formula (1.2), the integration over $d\omega$ must be extended between the limits $-\infty$ and $+\infty$. We shall integrate only over positive values of ω and so shall define the stress tensor as twice its usual expression. Thus

$$F=\int_0^\infty F_\omega\, d\omega=\frac{1}{4\pi}\int_0^\infty \{\overline{\mathbf{E}^2_{3r}}+\overline{\mathbf{H}^2_{3r}} \tag{2.1}$$

$$-\overline{E^2_{3x}}-\overline{H^2_{3x}}\}_{x=0}\, d\omega.$$

The dash over a symbol signifies a statistical averaging, to which the Fourier components **g** of the random field must be subjected. The averaging of components **g** referring to the same medium is carried out with the aid of Eq. (1.4) (with appropriate value of ϵ''). Quantities $\mathbf{g}_1$ and $\mathbf{g}_2$, referring to different media, are statistically independent, so the average of their products gives zero.

Writing the squares of the integrals (1.9) in the usual way as double integrals, and carrying out one integration over the δ - functions, we obtain after some transformations

$$F_\omega=\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_0^\infty\Big\{|v_y+w_y|^2+\frac{p^*}{p}|v_y-w_y|^2 \tag{2.2}$$

$$+\frac{c^2p^2}{\omega^2}|v_z+w_z|^2$$

$$+\frac{c^2|p|^2}{\omega^2}|v_z-w_z|^2\Big\}\,2\pi q\, dq\, dk_x,$$

where one must substitute in place of **v**, **w**, the expressions in the integrands of Eq. (1.13), and the average product $\overline{g_ig_k}$ is to be taken simply as $(A\epsilon''/4\pi^3)\,\delta_{ik}$. The integration over dk_x is carried out with the help of the formula

$$\int_{-\infty}^{+\infty}\frac{dk_x}{|k_x^2-s^2|^2}=\frac{i\pi}{|s|^2(s-s^*)}.$$

We replace the integration over dq by integration over dp, setting $q\,dq=p\,dp$.

After a sequence of transformations, we can represent F_ω in the following form:

$$F_\omega=\frac{\hbar}{4\pi^2}\operatorname{cth}\frac{\hbar\omega}{2T}$$

$$\times\int p^2\,dp\Big\{\Big[\frac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)}e^{-2ipl}-1\Big]^{-1}$$

$$+\frac{1}{2}\Big[\frac{(s_1+\varepsilon_1p)(s_2+\varepsilon_2p)}{(s_1-\varepsilon_1p)(s_2-\varepsilon_2p)}e^{-2ipl}-1\Big]^{-1} \tag{2.3}$$

$$+\frac{1}{2}\Big\}+\text{c.c.}$$

where c.c. denotes the complex conjugate expression, and the integration with respect to p is to be carried out in the plane of the complex variable p, over the segment $(\omega/c,\ 0)$ of the real axis and over the whole upper half of the imaginary axis.

It is an essential point that it turns out to be possible to represent F_ω as the real part of an integral of an analytic function of p, despite the fact that the expression (2.2) was obtained by taking square moduli of the field components. This can be done, if we note that on our integration path p is either pure real or pure imaginary. The integrand in Eq. (2.3) coincides with the integrand in Eq. (2.2), (after carrying out the k_x integration in the latter, and replacing $q\,dq$ by $p\,dp$), for just such values of p. But, having verified this, we can from now on regard this expression as an analytic function over the whole plane of the complex variable p, which enables us to make various transformations of the path of integration.

The expression (2.3) is itself finite, but contains terms which diverge upon integration over ω. These are the terms with ω^3, which appear as a result of the p integration of the terms with ½ in the curly brackets. However, this divergent term does not depend on the separation l of the bodies, and therefore has no connection with the problem of interest to us of the force of mutual attraction, and should be dropped. It represents the back reaction of the field produced by the body on the body itself, and is in fact compensated by similar forces on the other sides of the body.

For the following investigation of the integral (2.3), we change the notation, replacing p by $\omega p/c$ and s by $\omega s/c$. Also omitting the terms with ½, we have, finally,

$$F=\frac{\hbar}{2\pi^2c^3} \tag{2.4}$$

$$\times\operatorname{Re}\int_0^\infty\int p^2\omega^3\operatorname{cth}\frac{\hbar\omega}{2T}\Big\{\Big[\frac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)}e^{-2ip\omega l/c}-1\Big]^{-1}$$

$$+\Big[\frac{(s_1+\varepsilon_1p)(s_2+\varepsilon_2p)}{(s_1-\varepsilon_1p)(s_2-\varepsilon_2p)}e^{-2ip\omega l/c}-1\Big]^{-1}\Big\}\,dp\,d\omega,$$

$$s_1=\sqrt{\varepsilon_1(\omega)-1+p^2},$$

$$s_2=\sqrt{\varepsilon_2(\omega)-1+p^2}.$$

The paths of integration for p and ω are shown in Fig. 2a by the thick lines.

If we may consider the temperature of the bodies to be equal to zero (the necessary conditions for this will be explained later) then $\coth\dfrac{\hbar\omega}{2T}$ in Eq. (2.4) is replaced by unity. We shall first consider formula (2.4) for just this case.

Formula (2.4) is inconvenient because it is in complex form, and because the integrand contains the expression $e^{-2i\omega pl/c}$, which oscillates along

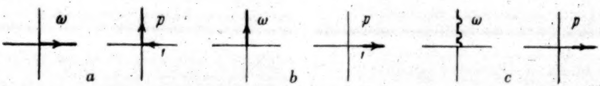

FIG. 2

the real part of the path of integration over p. The last fact makes the integration particularly difficult for large values of l, when the oscillation becomes very rapid. We can eliminate these difficulties by suitable changes of the paths of integration in the planes of the complex variables ω and p. Namely, we shall transform these paths so that the p integration is taken only over real, and the ω integration only over imaginary values; then the exponent in $e^{-2i\omega pl/c}$ will be real everywhere.

For the brevity, we shall denote the upper right quadrants of the ω and p planes (together with the semi-axes which bound them) as Q_ω and Q_p. We break up the path of integration over p in Eq. (2.4) into two parts, and first consider the one in which p runs through real values from unity to zero. Since we want to change the path of the ω integration from the positive real to the positive imaginary axis, we must investigate the question of the existence of singular points of the integrand as a function of ω in the region Q_ω.

According to the well-known general properties of the function $\epsilon(\omega)$, its imaginary part $\epsilon'' > 0$ everywhere in Q_ω except on the imaginary axis, where $\epsilon'' = 0$. On the latter, $\epsilon(\omega)$ is real and positive, decreasing monotonically from some value $\epsilon(0) > 1$ for $\omega = 0$ to unity for $\omega = i\infty$. Therefore the square root $s = \sqrt{\epsilon - 1 + p^2} = s' + is''$ (with real p) does not vanish anywhere within Q_ω, i.e., there are no branch points. From this it follows in turn that the inequalities $s' > 0$, $s'' > 0$, which are valid on the real semi-axis, are also valid everywhere within Q_ω.

The integrand might have poles at the roots of the denimonator in Eq. (2.4), i.e., the roots of the equations

$$\frac{(s_1 + p)(s_2 + p)}{(s_1 - p)(s_2 - p)} = e^{2ip\omega l/c}, \tag{2.5}$$

$$\frac{(s_1 + \epsilon_1 p)(s_2 + \epsilon_2 p)}{(s_1 - \epsilon_1 p)(s_2 - \epsilon_2 p)} = e^{2ip\omega l/c}.$$

But since s', s'', and ϵ'' are positive (in the region Q_ω) it is easy to see that for real p the moduli

$$\left|\frac{s + p}{s - p}\right| > 1, \quad \left|\frac{s + \epsilon p}{s - \epsilon p}\right| > 1,$$

and since $|e^{2ip\omega l/c}| \leq 1$, it is clear that Eqs. (2.5) can have no roots. Thus the integrand has no singularities in Q_ω. It also drops sufficiently rapidly at infinity, so the path of integration over ω can be shifted to the imaginary axis.

Next we turn to that part of the integral in which p runs through pure imaginary values from zero to $i\infty$. Here we must change both paths of integration, over p and over ω. However, we cannot make these changes in a simple succession, since, for example, it is impossible to show that in the general case there are no poles of the integrand in Q_ω for arbitrary imaginary values of p. But in integrating a function of several (in our case, two) complex variables, we have very great freedom in shifting the contours. Thus we can change from integration over certain contours C_ω and C_p to other paths C'_ω and C'_p, if we can in any way simultaneously shift the paths without having them pass through any singular points of the integrand. In the present case, such a procedure would be the simultaneous shift of the paths in the quadrants Q_ω and Q_p, during which the product $i\omega p$ remains real (and, clearly, negative):

$$\operatorname{Im}\{i\omega p\} = 0; \quad \operatorname{Re}\{i\omega p\} < 0. \tag{2.6}$$

In particular, the initial paths (real semi-axis for ω, imaginary semi-axis for p) and the final paths (imaginary semi-axis for ω, real semi-axis for p) satisfy this condition.

Such a transformation is , in fact, possible, for example, by introducing in place of the variable ω in the original integral the real positive variable $x = -i\omega p$, then shifting the integration over p from the imaginary to the real axis (for fixed values of x), and finally introducing ω once more as the imaginary quantity $\omega = ix/p$. Here, too, the integrand can have no branch points, since s could go to zero only for simultaneous pure imaginary values of ω and p, which event is excluded by condition (2.6). Therefore, we need only show that there are no roots of Eqs. (2.5) for values of p in the region Q_p for arbitrary real values of x. This presents no difficulties for the first of the equations, since the modulus $|\frac{s + p}{s - p}| \geq 1$ generally for all ω and p in Q_ω and Q_p, which is easily demonstrated by noting that $s' > 0$, $s'' > 0$.

The investigation is far more complicated for the second of Eqs. (2.5). We shall here outline the method of proof, assuming for simplicity that the two media are identical. From the equality:

$$\left(\frac{s+\varepsilon p}{s-\varepsilon p}\right)^2 = e^{-2xl}, \tag{2.7}$$

$$s = \sqrt{\varepsilon(ix/p) - 1 + p^2},$$

we conclude that $(s + \epsilon p)/(s - \epsilon p)$ must be a real number less than unity in absolute value. From this it follows in turn that there must be a relation between the values of the complex quantities s, p, ϵ of the form

$$s = -a\varepsilon p, \quad a > 0, \tag{2.8}$$

where a is some positive real number; one verifies easily that such a relation is possible only for $\epsilon' > 0$. This in turn excludes the possibility of having roots for values of x and p for which the argument of the function $\epsilon(ix/p)$ is very small or very large, since we know that $\epsilon' > 0$ in both these cases. There can also be no roots for very large values of p, since then $s \approx p$, and it would follow from (2.8) that $s = -1/a$, i.e., the value of s would be real and negative, which is impossible.

We shall first show that Eq. (2.7) has no roots for infinitesimal values of the parameter l. As we have pointed out, very large values of x are excluded for arbitrary l. For finite x, the right side of Eq. (2.7) approaches unity as $l \to 0$. The left side of the equation can tend to unity only if $p \to 0$, (since s does not go to zero anywhere). But then we would also have to have $x \to 0$ (in order that the ratio x/p remain finite), and the right side of Eq. (2.7) will approach unity for $l \to 0$ faster than the left side, so that for sufficiently small l, there can be no roots in any case.

We show further that for arbitrary values of l there are no roots on the boundaries of the region Q_p[‡]. In fact, infinitely large values of p are excluded as shown above, while the axis of abscissas (real p) is out, since the function $\epsilon(ix/p)$ of imaginary argument is real and positive. The ordinate axis (imaginary p) is excluded, since for such p the relation (2.8), when squared, would give a quadratic equation for ϵ:

$$\epsilon^2 a^2 \mid p \mid + \epsilon - (1 + \mid p \mid^2) = 0$$

from which it would follow that ϵ were real, which is impossible (for real argument ix/p).

With increasing l, roots could steal into the region Q_p only across its boundaries. Therefore, the absence of roots for very small l, and the fact that there are no roots on the boundaries of Q_p for arbitrary l, shows that there are no roots anywhere in Q_p for arbitrary values of l[+].

Thus the required change of integration path can be carried out in both parts of the integral. Upon adding the two parts, the integrals over p from zero to unity cancel, and we obtain the following expression for the force of interaction for $T = 0$):

$$F = \frac{\hbar}{2\pi^2 c^3} \times \int_0^\infty \int_1^\infty p^2 \xi^3 \left\{ \left[\frac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)} e^{2p\xi l/c} - 1 \right]^{-1} + \left[\frac{(s_1+p\varepsilon_1)(s_2+p\varepsilon_2)}{(s_1-p\varepsilon_1)(s_2-p\varepsilon_2)} e^{2p\xi l/c} - 1 \right]^{-1} \right\} dp\, d\xi \tag{2.9}$$

(the path of integration is shown in Fig. 2b). Here we have introduced the notation $\omega = i\xi$ for imaginary values of ω, and ϵ_1 and ϵ_2 are to be taken as the real functions $\epsilon_1(i\xi)$ and $\epsilon_2(i\xi)$. We have dropped the "Re", since the expression given is manifestly real. Formula (2.9) makes it possible, in principle, to compute the force F for any separation l, if only the functions $\epsilon(i\xi)$ are known for both bodies. The latter can be expressed in terms of the value of the imaginary part of the function $\epsilon(\omega)$ for real ω by

$$\varepsilon(i\xi) - 1 = \frac{2}{\pi} \int_0^\infty \frac{\omega\varepsilon''(\omega)}{\omega^2 + \xi^2} d\omega. \tag{2.10}$$

Thus we may say that the law of interaction of bodies is determined if we give the functions $\epsilon''(\omega)$; (we shall see, in Sec. 5, that this remains true for temperatures different from zero).

3. THE CASE OF SMALL SEPARATIONS

We first consider the limiting case of distances l which are small compared to the wavelengths that are important in the absorption spectra of the bodies (for a more exact formulation of this condition, see below). The temperatures which can occur for condensed bodies are always small compared to the values $\hbar\omega$ (e.g., in the visible region of the spectrum) which are important here. Therefore we may set $T \approx 0$, and use formula (2.9).

[‡] Except for the irrelevant, trivial root $p = 0$ for $x = 0$, whose position does not depend on l.

[+] The possibility of a root sneaking in at the point $\omega = 0$ is also excluded: for this to occur there would have to be a double root at $\omega = 0$ for some value of l, which does not occur.

Because of the presence of the exponentially increasing factor $e^{2p\xi l/c}$ in the denominators of the integrands, the main contributions to the integral over p come from values of p, for which $p\xi l/c \sim 1$. But then $p \gg 1$, so that, in finding the leading terms, we can set $s_1 \approx s_2 \approx p$. In this approximation, the first term in square brackets in Eq. (2.9) becomes zero. After introducing the integration variable $x = 2lp\xi/c$, the second term gives*

$$F = \frac{\hbar}{16\pi^2 l^3}\int_0^\infty\int_0^\infty \frac{x^2 dx\, d\xi}{\left(\frac{\varepsilon_1+1}{\varepsilon_1-1}\right)\left(\frac{\varepsilon_2+1}{\varepsilon_2-1}\right)e^x-1} \qquad (3.1)$$

(in this approximation, the lower limit $2l\xi/c$ of the x integration is set equal to zero).

Formula (3.1) determines the force of attraction in the limiting case of small l. The force turns out to be inversely proportional to the cube of the separation, which, as was to be expected, is in accordance with the usual van der Waals force between two atoms (neglecting retardation). With increasing ξ, the functions $\epsilon(i\xi) - 1$ decrease monotonically to zero. Thus, starting with some $\xi \sim \xi_0$, larger values of ξ cease giving any significant contribution to the integral; the condition on the smallness of l is that $l \ll c/\xi_0$.

Let us show how the transition to the limit of interaction of individual atoms is carried out in Eq. (3.1). For this purpose, we assume that both media are sufficiently rarefied. Then the differences $\epsilon_1 - 1$ and $\epsilon_2 - 1$ are close to zero, and we have, from Eq. (3.1), to sufficient accuracy,

$$F = \frac{\hbar}{64\pi^2 l^4}\int_0^\infty\int_0^\infty x^2 e^{-x}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\, dx\, d\xi$$

$$= \frac{\hbar}{32\pi^2 l^3}\int_0^\infty [\varepsilon_1(i\xi) - 1][\varepsilon_2(i\xi) - 1]\, d\xi.$$

Expressing $\epsilon(i\xi)$ in terms of the values of $\epsilon''(\omega)$ on the real axis of ω, in accordance with Eq. (2.10), we obtain

$$\int_0^\infty [\varepsilon_1(i\xi) - 1][\varepsilon_2(i\xi) - 1]\, d\xi$$

$$= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\omega_1\omega_2\varepsilon_1''(\omega_1)\varepsilon_2''(\omega_2)}{(\omega_1^2+\xi^2)(\omega_2^2+\xi^2)}\, d\xi\, d\omega_1\, d\omega_2$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\varepsilon_1''(\omega_1)\varepsilon_2''(\omega_2)}{\omega_1+\omega_2}\, d\omega_1\, d\omega_2,$$

and find for the force F

$$F = \frac{\hbar}{16\pi^3 l^3}\int_0^\infty\int_0^\infty \frac{\varepsilon_1''(\omega_1)\varepsilon_2''(\omega_2)}{\omega_1+\omega_2}\, d\omega_1\, d\omega_2. \qquad (3.2)$$

This force corresponds to an interaction of the atoms with energy

$$U = -\frac{3\hbar}{8\pi^4 R^6 N^2} \qquad (3.3)$$

$$\times \int_0^\infty\int_0^\infty \frac{\varepsilon_1''(\omega_1)\varepsilon_2''(\omega_2)}{\omega_1+\omega_2}\, d\omega_1\, d\omega_2,$$

where N is the number of atoms per unit volume. [Equation (3.2) is obtained from Eq. (3.3) by integrating over both half-spaces and then differentiating with respect to the distance l between them]. Formula (3.3) agrees exactly with the well-known formula of London[5], obtained by applying ordinary perturbation theory to the dipole interaction of two atoms. In making the comparison, one must note that the imaginary part of $\epsilon(\omega)$ is related to the spectral density $f(\omega)$ of "oscillator strengths" by the relation

$$\omega\varepsilon''(\omega) = (2\pi^2 e^2/m)\, Nf;$$

while the oscillator strengths are themselves expressed as usual in terms of the squares of the matrix elements of the dipole moment of the atom.

Formula (3.1) can be represented sufficiently accurately for all practical purposes in a simple form. Let us assume, for brevity, that both bodies are identical. The integral with respect to x, in Eq. (3.1), depends, aside from the parameter, on the quantity $[(\epsilon + 1)/(\epsilon - 1)]^2$, which takes on values never less than unity (unity is reached for $\epsilon \to \infty$). Figure 3 shows a graph of the integral

$$I = \frac{a}{2}\int_0^\infty \frac{x^2\, dx}{ae^x - 1}$$

as a function of the parameter a. This integral tends to unity for $a \to \infty$, but we see that even for $a = 1$ it differs from unity by at most 20% and that this difference drops rapidly with increasing a. So we can

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* This same result could have been gotten directly from Eq. (2.4), with the paths of integration shown in Fig. 2a, by noting that the main contribution to the integral comes from imaginary values of p. As already noted, imaginary values of p correspond to exponentially damped ("inhomogeneous") plane waves. It is entirely natural that just this part of the fluctuation field (and not the undamped, true plane waves) gives the main contribution to the interaction force for separations at which retardation effects are still unimportant.

[5] R. Eisenschitz and F. London, Z. Physik 60, 491 (1930)

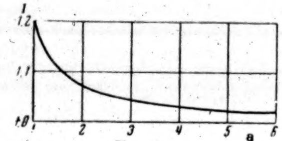

FIG. 3

practically write Eq. (3.1) in the form

$$F = \frac{\hbar}{8\pi^2 l^3} \int_0^\infty \left(\frac{\varepsilon(i\xi)+1}{\varepsilon(i\xi)-1}\right)^2 d\xi \tag{3.4}$$

(with a similar formula if the bodies are different).

To evaluate the accuracy of the limiting law of interaction which we have obtained, it is useful to have the next term in the expansion of the function $F(l)$. A calculation using the same general formula (2.9) gives (for identical bodies) the expression

$$-\frac{\hbar}{8\pi^2 c^2 l} \int_0^\infty \frac{\xi^2 [\varepsilon(i\xi)-1]^2}{\varepsilon(i\xi)+1} d\xi, \tag{3.5}$$

which should be added to Eq. (3.4). However, it is impossible to give a concrete evaluation of the range of applicability of the limiting law without specific knowledge of the function $\epsilon(i\xi)$.

4. THE CASE OF LARGE SEPARATIONS

We now go over to the opposite limiting case of distances which are large compared to the fundamental wavelengths in the absorption spectrum of the bodies. Once again we shall first take the temperature to be equal to zero; for the meaning of this approximation in the present case, see below.

Again we introduce a new integration variable $x = 2pl\xi/c$ in the general formula, but we now keep as our second variable not ξ (as in Sec. 3), but rather p:

$$F = \frac{\hbar c}{32\pi^2 l^4} \int_0^\infty \int_1^\infty \frac{x^3}{p^2} \left\{ \left[\frac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)} e^x - 1 \right]^{-1} + \left[\frac{(s_1+\varepsilon_1 p)(s_2+\varepsilon_2 p)}{(s_1-\varepsilon_1 p)(s_2-\varepsilon_2 p)} e^x - 1 \right]^{-1} \right\} dp\,dx, \tag{4.1}$$

$$\varepsilon = \varepsilon\left(i\frac{xc}{2pl}\right), \quad s = \sqrt{\varepsilon\left(i\frac{xc}{2pl}\right) - 1 + p^2}.$$

Because of the presence of e^x in the denominators, the main contribution to the x integral comes from $x \sim 1$, so since $p \geq 1$, the argument of the function ϵ, for large l, is close to zero over the whole important range of values of the variables. In accordance with this, we may simply replace ϵ_1 and ϵ_2 by their values for $\omega = 0$, i.e., by the static dielectric constants, which we denote by ϵ_{10}, ϵ_{20}. We know that for metals the function $\epsilon(\omega)$ tends toward infinity as $\omega \rightarrow 0$; in this case we must set $\epsilon_0 = \infty$. In this way we finally obtain the following result:

$$F = \frac{\hbar c}{32\pi^2 l^4} \int_0^\infty \int_1^\infty \frac{x^3}{p^2} \left\{ \left[\frac{(s_{10}+p)(s_{20}+p)}{(s_{10}-p)(s_{2},-p)} e^x - 1 \right]^{-1} + \left[\frac{(s_{10}+\varepsilon_{10} p)(s_{20}+\varepsilon_{20} p)}{(s_{10}-\varepsilon_{10} p)(s_{20}-\varepsilon_{20} p)} e^x - 1 \right]^{-1} \right\} dp\,dx, \tag{4.2}$$

$$s_{10} = \sqrt{\varepsilon_{10} - 1 + p^2}, \quad s_{20} = \sqrt{\varepsilon_{20} - 1 + p^2}.$$

Here the force of attraction is inversely proportional to l^4. It should be noted that in this limiting case it depends only on the static dielectric constants of the two media.

The integration with respect to p in Eq. (4.2) can be carried out in terms of elementary functions, after which there remains a single integral over x, depending on the two constant parameters ϵ_{10} and ϵ_{20}. We shall not give the corresponding, very complicated, general expressions, the more so since for purposes of numerical integration it is clearly more convenient to start directly from the double integral of Eq. (4.2).

Let us consider some special cases. In particular, a simple result is obtained for two metals. Setting $\epsilon_{10} = \epsilon_{20} = \infty$, we get

$$F = \frac{\hbar c}{16\pi^2 l^4} \int_0^\infty \int_1^\infty \frac{x^3 dp\,dx}{p^2(e^x - 1)} = \frac{\hbar c}{l^4} \frac{\pi^2}{240}. \tag{4.3}$$

This force does not depend in any way on the nature of the metals [which was not the case for small separations (Sec. 3), where the magnitude of the interaction depended on the function $\epsilon(i\xi)$ for all values of ξ, and not just at $\xi = 0$]. Formula (4.3) coincides with the formula, obtained by Casimir[6], for this special case, by considering the normal modes of the field in the gap between two walls which are ideally reflecting at all frequencies.

For two identical dielectrics ($\epsilon_{10} = \epsilon_{20} = \epsilon_0$) we give the result obtained from Eq. (4.2) by numerical integration:

$$F = \frac{\hbar c}{l^4} \frac{\pi^2}{240} \left(\frac{\varepsilon_0 - 1}{\varepsilon_0 + 1}\right)^2 \varphi(\varepsilon_0), \tag{4.4}$$

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[6] H. B. G. Casimir, Proc. Nederl. Akad. Wetensch., 51, 793 (1948)

where $\varphi(\epsilon_0)$ is a function whose values are shown in the graph of Fig. 4 (the curve *DD*). For $\epsilon_0 \to \infty$, this function approaches unity according to the law

$$\varphi(\varepsilon_0) = 1 - \frac{1.11}{\sqrt{\varepsilon_0}} \ln \frac{\varepsilon_0}{7.6} \tag{4.5}$$

(Note that this formula is accurate only for very large values of ϵ_0). For $\epsilon_0 \to 1$, the function $\varphi(\epsilon_0)$ approaches a finite limit, 0.35, corresponding to the limiting law (4.7) (see below). This limit is, however, practically reached for $\epsilon_0 \approx 4$, after which $\varphi(\epsilon_0)$ remains practically constant.

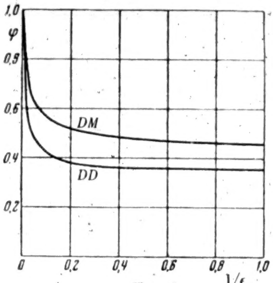

FIG. 4

In the same Fig. 4 we show the curve (*DM*) of the similar function which gives the force of attraction between a metal and a dielectric ($\epsilon_{10} = \infty$, $\epsilon_{20} = \epsilon_0$) according to the formula:

$$F = \frac{\hbar c}{l^4} \frac{\pi^2}{240} \frac{\varepsilon_0 - 1}{\varepsilon_0 + 1} \varphi(\varepsilon_0). \tag{4.6}$$

Finally we carry out the transition to interaction of individual atoms in Eq. (4.2). To do this, we assume as in Sec. 3 that both media are sufficiently rarefied, i.e., that the differences $\epsilon_{10} - 1$ and $\epsilon_{20} - 1$ are small.

Keeping only the first non-vanishing terms in the expansion of the integrands of Eq. (4.2) in powers of these differences, we get

$$F = \frac{\hbar c}{32\pi^2 l^4} (\varepsilon_{10} - 1)(\varepsilon_{20} - 1) \times \int_0^\infty x^3 e^{-x} dx \int_1^\infty \frac{1 - 2p^2 + 2p^4}{8p^6} dp$$

or

$$F = \frac{\hbar c}{l^4} \frac{23}{640\pi^2} (\varepsilon_{10} - 1)(\varepsilon_{20} - 1). \tag{4.7}$$

This force corresponds to interaction of the atoms with energy

$$U = -\frac{23\hbar c}{64\pi^3 R^7} \frac{(\varepsilon_{10} - 1)(\varepsilon_{20} - 1)}{N^2} = -\frac{23\hbar c}{4\pi R^7} \alpha_1 \alpha_2, \tag{4.8}$$

where α_1, α_2 are the static polarizabilities of the two atoms. This formula coincides with the results of reference 1 for the van der Waals forces, including retardation effects; we have here obtained it from macroscopic considerations.

To estimate the accuracy and range of validity of the formulas obtained, we must again, as in Sec. 3, find the next term in the expansion of the function $F(l)$[+]. We shall do this for the case of two metals, which we assume to be identical.

Formula (4.3) is gotten from (4.1) if we set $\epsilon_1 = \epsilon_2 = \infty$ in the latter. But if we also want to get the next term in the expansion, we must use that form for the function $\epsilon(\omega)$ which is valid in the frequency region which is important in the integration. As we have seen, the important region is $\omega/c \sim 1/l$, i.e., $\lambda \sim l$. Accordingly, we set

$$\varepsilon(\omega) = -4\pi e^2 N / m\omega^2, \tag{4.9}$$

where N is the number density of free electrons in the metal; this formula, which is sufficiently good for our purposes, gives the general behavior of $\epsilon(\omega)$ in the infrared region of the spectrum[‡]. When substituting in Eq. (4.1), we must replace ω by $icx/2pl$; then expanding the integrand in powers of $1/l$, we obtain

$$F = \frac{\hbar c}{32\pi^2 l^4} \left\{ \frac{2\pi^4}{15} - \frac{c}{el} \sqrt{\frac{m}{\pi N}} \times \int_0^\infty \frac{x^4 e^x dx}{(e^x - 1)^2} \int_1^\infty \frac{p^2 + 1}{p^4} dp \right\},$$

from which, finally,

$$F = \frac{\hbar c}{l^4} \frac{\pi^2}{240} \left\{ 1 - 7{,}2 \frac{c}{el} \sqrt{\frac{m}{N}} \right\}. \tag{4.10}$$

[+] If we set $\epsilon(i\xi) = 1 + a/(\omega_0^2 + \xi^2)$ (so that $\epsilon_0 = 1 + a/\omega_0^2$), then calculation of F from formulas (3.4) and (4.4) shows that the two values match for $l \cong c/\omega_0$. This computation allows us to conclude that the characteristic length for comparison with l is not the wavelength in the absorption region, but rather $\lambda/2\pi$.

[‡] For still larger values of λ, the function $\epsilon(\omega)$ goes over into $\epsilon = 4\pi i\sigma/\omega$, where σ is the ordinary electrical conductivity of the metal. However, the corresponding frequency region gives a very small contribution to the integral.

So, tentatively setting $N = 5.9 \times 10^{22}\text{cm}^{-3}$ (for silver), we find that the second term is small compared to the first, if $l \gg 0.6\ \mu$.

We note that the second term in the expansion, which we have found here, could not be obtained by the method applied in reference 6 for getting the leading term.

5. THE EFFECT OF TEMPERATURE ON THE FORCE OF INTERACTION

While we can practically always consider the temperature of the bodies to be equal to zero for the limiting case of small separations (sec. 3), the effect of temperature may be substantial for large separations. Anticipating later results, we state that the condition for setting $T = 0$ is roughly speaking, $lT/\hbar c \ll 1$. For sufficiently low temperatures, this condition will of course always be compatible with the condition determining the lower bound of values of l for which the limiting law obtained in Sec. 4 is valid. But these two conditions may, as for example at room temperature, turn out to be incompatible; then the region in which the limiting laws obtained in Sec. 4 are applicable actually does not exist.

To obtain the formulas including the effect of temperature, we turn to the original expression (2.4) and see how, for $T \neq 0$, we must change the transformations which led to formula (2.9) in the case of $T = 0$. The function $\coth \hbar\omega / 2T$ has an infinite number of poles, located on the imaginary axis, and equal to

$$\omega_n = i\xi_n = i\,\frac{2\pi T}{\hbar}\,n, \tag{5.1}$$

where n is an integer. Therefore, upon shifting the path of integration to the imaginary axis, we must go around these poles on semicircles (as shown in Fig. 2c). These circuits give contributions to the real part of the integral, which are equal to $i\pi$ times the residue of the integrand at the pole. (The integration over the segments of the imaginary axis between the poles gives a pure imaginary number, which drops out when we take the real part of the expression).

The point with $n = 0$ ($\omega = 0$) requires special consideration. At first glance it might appear that this point is not a pole of the integrand in the integral over ω in (2.4), because of the presence of the factor ω^3. However, this factor vanishes upon integration over p [cf. also the expression (2.3) for F_ω]. The presence of a pole at the point $\omega = 0$ does not of course lead to divergence of (2.4), since for $\omega \to 0$ along the real axis the divergent contribution to the integral is pure imaginary, and drops out when we take the real part. [This can be seen more clearly from the expression (2.3) for F_ω, which remains finite for $\omega = 0$].

To take this point into account when transforming the path of integration, we shall suppose that the integration over ω in Eq. (2.4) is carried from some sufficiently small δ to ∞ (and not from zero to infinity); as we showed above, the real part of the integral is not changed when we do this. Upon shifting the contour to the imaginary axis, a circuit is added along a quarter-circle around the point $\omega = 0$ (Fig. 2c). This circuit gives a contribution to the integral equal to $i\pi/2$ times the corresponding residue.

To simplify writing of formulas, we shall assume the bodies to be identical; generalization to different bodies, on the basis of the general form of Eq. (2.4), is obvious.

We thus obtain the following formula:

$$F = \frac{T}{\pi c^3} {\sum_{n=0}^{\infty}}' \int_1^\infty p^2 \xi_n^3 \left\{ \left[\left(\frac{s_n + p}{s_n - p} \right)^2 e^{2p\xi_n l/c} - 1 \right]^{-1} + \left[\left(\frac{s_n + \varepsilon_n p}{s_n - \varepsilon_n p} \right)^2 e^{2p\xi_n l/c} - 1 \right]^{-1} \right\} dp,$$

$$s_n = \sqrt{\varepsilon_n - 1 + p^2}, \quad \varepsilon_n = \varepsilon(i\xi_n). \tag{5.2}$$

The prime on the summation sign means that the term with $n = 0$ should be taken with a factor ½. Replacing p by the integration variable $x = pn$, we rewrite (5.2) in the form:

$$F = \frac{8\pi^2 T^4}{\hbar^3 c^3} {\sum_{n=0}^{\infty}}' \int_n^\infty x^2 \left\{ \left[\left(\frac{ns_n + x}{ns_n - x} \right)^2 e^{4\pi l T x/\hbar c} - 1 \right]^{-1} + \left[\left(\frac{ns_n + \varepsilon_n x}{ns_n - \varepsilon_n x} \right)^2 e^{4\pi l T x/\hbar c} - 1 \right]^{-1} \right\} dx, \tag{5.3}$$

$$ns_n = \sqrt{n^2(\varepsilon_n - 1) + x^2}, \quad \varepsilon_n = \varepsilon\left(i\,\frac{2\pi T}{\hbar}\,n \right).$$

Formula (5.2) or (5.3) enables us, in principle, to calculate the force F for any value of l and any temperature. We see that, for $T \neq 0$ also, it is sufficient to know the values of the function $\epsilon(i\xi)$.

For $T \to 0$, the distances between poles also tend to zero, the summation over n can be replaced by an integration over ξ, and we return to formula (2.6) which does not contain T. By determining the first correction to this formula, we can establish a criterion for setting $T = 0$ in calculating the attractive force. We shall do this for metals, applying formula (4.9) for $\epsilon(\omega)$ as we did in Sec. 4. According to the Euler sum formula we have, for a function $f(n)$ which together with all its derivatives goes to zero for $n \to \infty$,

$$\sum_{n=0}^{\infty}{}' f(n) = \int_0^{\infty} f(n)\,dn + \frac{1}{12} f'(0) - \frac{1}{30 \cdot 4!} f'''(0) + \ldots$$

In our case the function $f(n)$ is the integral under the summation sign in (5.3). In the calculation we shall assume that l is small compared to $\hbar c/T$, but still large compared to the quantity $(c/e)\sqrt{m/N}$ which is characteristic for the metal. Then $f'(0) = 0$, $f'''(0) = 2$ and thus

$$F = \frac{\pi^2}{240}\frac{\hbar c}{l^4}\left[1 - \frac{48}{9}\left(\frac{lT}{\hbar c}\right)^4\right]. \qquad (5.4)$$

Thus, at room temperature the correction is already small if $l < 5\,\mu$; comparison with the criterion obtained in Sec. 4 shows that in this case there is a region in which the formulas there obtained are applicable*.

In the opposite limiting case of large values of $lT/\hbar c$, we need keep only the first term in the sum in Eq. (5.3), i.e., $n = 0$:

$$F = \frac{4\pi^2 T^4}{\hbar^3 c^3} \times \int_0^{\infty} \frac{x^2\,dx}{[(\varepsilon_0 + 1)/(\varepsilon_0 - 1)]^2 \exp\left\{\frac{4\pi l T}{\hbar c} x\right\} - 1}$$

or

$$F = \frac{T}{16\pi l^3}\int_0^{\infty}\frac{x^2\,dx}{[(\varepsilon_0 + 1)/(\varepsilon_0 - 1)]^2 e^x - 1} \qquad (5.5)$$

$$\approx \frac{T}{8\pi l^3}\left(\frac{\varepsilon_0 - 1}{\varepsilon_0 + 1}\right)^2.$$

Thus for sufficiently large separations, the interaction force stops dropping so rapidly, and once again follows a $1/l^3$ law, with a coefficient which depends on the temperature and on the static value of the dielectric constant. This fact has apparently not been previously noted anywhere in the literature.

All the other terms in the sum (5.3) decrease exponentially for large $lT/\hbar c$. So, including the first correction term, we obtain for two metals

$$F = \frac{T}{8\pi l^3}\left[1 + 2\left(\frac{4\pi T l}{\hbar c}\right)^2 \exp\{-4\pi T l/\hbar c\}\right]. \qquad (5.6)$$

Let us say a few words about the comparison of the results of the theory presented in this paper with experiment. Direct measurements of molecular attractive forces are very difficult, and apparently the only work in which the authors have succeeded in eliminating all spurious effects is that of Abrikosova and Deriagin[7]. These authors measured the attractive force between quartz plates for separations 0.1 - 0.4 μ. Exact comparison with theory would require sufficiently complete knowledge of the optical characteristics of the material over its absorption regions; without this we cannot construct the function $\epsilon(i\xi)$. However, the character of the absorption in quartz enables us to make an approximate theoretical estimate**. Considering the crudeness of this estimate, and possible errors in the measurement, we may state that the agreement between the theory and the experimental data is satisfactory.

In conclussion I express my sincere thanks to Academician L. D. Landau for discussion of the problems considered here. I also thank I. I. Abrikosova and B. V. Deriagin for discussion of the experimental data, and I. G. Krutikova who carried out the numerical calculations mentioned in the text.

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* With the function $\epsilon = 4\pi i\sigma/\omega$ (cf footnote, p. 81) we would get a very much greater upper limit for l.

[7] I. I. Abrikosova and B. V. Deriagin, Dokl. Akad. Nauk SSSR, **90**, 1055 (1953); I. I. Abrikosova. Dissertation, Phys. Chem. Inst., Academy of Sciences USSR, 1954.

** *Note added in proof.*- Quartz has strong absorption in the ultraviolet (starting at about 0.15μ) and in the infrared (starting at a few μ), between which regions it is transparent. The separations used in the experiments fall in the region of transparency; in making the estimate we may assume that l is small compared to $\lambda/2\pi$ (cf footnote, p.   ) for the right absorption edge, and large compared to the left absorption edge. The contribution of the ultraviolet absorption region to the force F can be evaluated from formula (4.4), setting ϵ_0 equal to the square of the refractive index in the region of optical transparency. The contribution of the infrared region is given by formula (3.4); in order of magnitude it is a factor $l\omega_0/c$ smaller (ω_0 is the infrared absorption frequency), and can be neglected in a rough estimate of F. Thus we obtain for the force a $1/l^4$ law, with a coefficient determined as above. This estimate is raised for larger separations and lowered for smaller separations.

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The Theory of Λ^0 Particles

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A class of equations possessing internal degrees of freedom is investigated from the formal point of view for its possibilities in describing the Λ^0 particle as an excited state of the nucleon.

IN his works, I. E. Tamm repeatedly returned to the idea of excited nucleon states[1].

Discoveries of new particles in recent years (hyperons, heavy mesons) make natural further investigations of different possibilities for introducing some kind of internal degrees of freedom for the elementary particles. The discovery of a hyperon heavier than the Λ^0 particle --- the X hyperon --- and the latter's disintegration cascade $X \to \pi^- + \Lambda^0 \to \pi^- + p + \pi^-$ makes plausible, if not conclusive, the hypothesis that the hyperons can be represented as excited states of nucleons.

We shall investigate the possibility of one such hyperon model from this point of view.

One of us has made an attempt to write the equations of an elementary particle with an increasing mass spectrum[2]. One of these equations has the form

$$\Big\{ \gamma_\mu k_\mu + m_0 + a\Big[-\frac{\partial^2}{\partial\xi_\mu \partial\xi_\mu} + \xi_\mu\xi_\mu + 2\,\frac{\left(k_\mu \frac{\partial}{\partial\xi_\mu}\right)^2 - (k_\mu\xi_\mu)^2}{k_\mu k_\mu.}\Big]^n + g\gamma_5\tau_i\varphi_i(x+r)\Big\}\psi = 0, \tag{1}$$

where $\xi_\mu = r_\mu / r_0$ is the dimensionless internal degree of freedom, r_0 is the length constant, x is the "center of mass" coordinate of the particle, k_μ is its momentum, φ is the pseudoscalar meson field, and n is an integer.

Equation (1) is outside the framework of the class of equations investigated in reference 3.

We shall regard the Λ^0 particle then as an excited state of the nucleon. We should like to state clearly that our purpose is merely to discuss the possibilities of a class of similar equations, and that this investigation at the present time might have only a formal significance. The selection of a definite form of Eq. (1) itself is associated with a wide freedom of choice.

For simplicity we shall let $n = 1$ in Eq. (1).

Now the following properties are characteristic of the Λ^0 particle:

a) The disintegration of the Λ^0 particle into a proton and a π^- meson:

$$\Lambda^0 \to p + \pi^- + (Q = 35 \quad \text{mev}). \tag{2}$$

b) A long lifetime: $\tau \sim 3 \times 10^{-10}$ sec.

c) A Λ^0 particle, apparently, may be found in a complex nucleus in a bound state.

d) The creation of Λ^0 particles in a gas from Θ^0 particles.

e) A relatively high probability for the creation of Λ^0 particles in the reaction $\pi^- + p \to \Lambda^0 + \Theta^0$.

We shall investigate the disintegration of a Λ^0 particle at rest. The constant a in Eq. (1) is determined from the observation a) for the disintegration of Λ^0: $a = \mu_\pi + Q \sim 180$ mev. It characterizes the energy interval between two levels of the excited nucleon.

The unperturbed wave function of Eq. (1) has the form

$$\Psi = U_m(x)\,\chi_{n_1 n_2 n_3 n_4}(k_1\xi), \tag{3}$$

where $U_m(x)$ is the solution of the usual Dirac equation in the form of a plane wave corresponding to a mass m. The function χ for the unexcited state of the nucleon, for example, has the form

$$\chi_{0000} = \text{const}\exp\Big\{-\Big(\frac{\xi_\mu\xi_\mu}{2} + \frac{(k_\mu\xi_\mu)^2}{m^2}\Big)\Big\}, \tag{4}$$

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[1] V. L. Ginzburg and I. E. Tamm, J. Exper. Theoret. Phys. USSR 17, 227 (1947)

[2] M. A. Markov, Dokl. Akad. Nauk SSSR 101, 51 (1955)

[3] I. M. Gel'fand and A. Iaglom, J. Exper. Theoret. Phys. USSR 18, 703 (1948)

where $\xi_\mu \xi_\mu = \xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_0^2$; $n_1 = n_2 = n_3 = n_0 = 0$.

In the non-relativistic approximation the function (4) becomes

$$\chi_{0000} = \text{const} \exp\{-(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_0^2)\}. \quad (5)$$

The mass operator in Eq. (1) in this case admits the simple expression

$$\left\{ \ldots m_0 + a \left\{ \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_0^2 - \frac{\partial^2}{\partial \xi_1^2} \right.\right. \quad (6)$$

$$\left.\left. - \frac{\partial^2}{\partial \xi_2^2} - \frac{\partial^2}{\partial \xi_3^2} - \frac{\partial^2}{\partial \xi_0^2} \right\} \Psi = 0, \right.$$

That is, in the rest system we have a four-dimensional oscillator in the literal sense of the word, inasmuch as the real time component ξ_0 enters in Eqs. (5) and (6) with the same sign as the other three components.

Let us consider initially the Λ^0 particle as the first excited state of the nucleon. The internal wave function χ_{1000} of the Λ^0 particle at rest, for example, assumes the form

$$\chi_{1000} = \frac{1}{\pi r_0^2} \sqrt{2}\, \xi_1 \quad (7)$$

$$\times \exp\left\{-\frac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_0^2)\right\}.$$

In the present variation of the theory, four different states of the Λ^0 particle are possible: χ_{1000}, χ_{0100}, χ_{0010}, χ_{0001}.

In the variation described by Eq. (15)* the state χ_{0001} is excluded.

Neglecting the recoil of the proton upon disintegration of the particle at rest, we obtain for the final state of the proton the internal function

$$\chi_{0000} = \frac{1}{\pi r_0^2} \exp\left\{-\frac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_0^2)\right\}. \quad (8)$$

We shall write the interaction, under whose influence the transition from state (7) to state (8) takes place, in the form

$$H' = g\gamma_5 \frac{\hbar c}{\sqrt{2E_\pi V}} \exp\{-i(k_\mu^\pi \xi_\mu r_0 + k_\mu^\pi x_\mu\}. \quad (9)$$

The matrix element corresponding to the reaction $\Lambda^0 \to p + \pi^-$ will be written in the following manner:

$$M_{01} = \sim g \frac{(U_p^* \beta\gamma_5 U_{\lambda 0}(k_1 r_0)}{\sqrt{E_\pi}} \quad (10)$$

$$\times \exp\left\{-\frac{r_0^2}{4}(k_1^2 + k_2^2 + k_3^2 + k_0^2)\right\},$$

where $\hbar k_i$ is the component of the momentum of the π^- meson (or proton). The form-factor in Eq. (10) is retained by us in the non-relativistic approximation. This factor would have the relativistic form

$$\exp\left\{-\frac{r_0^2}{4}\left(k_\mu^\pi k_\mu^\pi + 2\frac{(k_\nu^p k_\nu^\pi)^2}{k_\mu^p k_\mu^p}\right)\right\}; \quad (10')$$

If the momentum of the proton $\hbar k^p = 0$, and $\hbar k_0^p = m_p$, we obtain the form-factor (10). For the probability of the process discussed we obtain the expression

$$\omega = \frac{1}{\tau} \sim \frac{g^2}{\hbar c} \frac{P^2}{m_p^2 c^2} \left(\frac{P r_0}{\hbar}\right)^2 \frac{Pc}{\hbar} \quad (11)$$

$$\times \exp\left\{-\frac{r_0^2}{2\hbar^2}(P_1^2 + P_2^2 + P_3^2 + P_0^2)\right\},$$

where P is the momentum of the decay proton (or meson). The multiplier $P^2/m_p^2 c^2$ in Eq. (11) appears as a consequence of the assumed pseudo-scalar interaction; for scalar mesons it is replaced by unity.

From the given lifetime of the Λ^0 particle it is possible to determine the second constant in Eq. (1), the "sizes" r_0 of the oscillator, assuming that the relative smallness of ω in Eq. (11) is completely a consequence of the smallness of r_0. Putting $k \sim \mu_\pi c$ $(k_i \sim \mu_\pi c/\sqrt{3})$, $m_p \sim 6_{\mu_\pi}$, we obtain for $r_0^{(1)}$ the estimate

$$r_0^{(1)} \sim \frac{\hbar}{\mu_\pi c} \left(\frac{\hbar}{\mu_\pi c c \tau}\right)^{1/3} \sqrt{\frac{\hbar c}{g^2}}; \quad (12)$$

Assuming that $(\hbar c/g^2)^{1/2} \sim 1$, we obtain

$$r_0 \sim 10^{-19} \text{ cm}. \quad (13)$$

Let us now suppose that the Λ^0 particle is associated with the *n*th excited state of a nucleon. Then, using similar crude estimates, we obtain

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* Taking into account the recoil gives certain additional possibilities for the formation of short-lived excited states with small spin and long-lived ones with large spin. The point is that if the recoil is taken into account, it turns out that transitions from the excited to the ground state are allowed if $n_0 = n_1 + n_2 + n_3$. We are indebted for this observation to L. G. Zastavenko.

$$r_0^{(n)} \sim \frac{\hbar}{p}\left(\frac{\hbar}{pc\tau}\,\frac{\hbar c}{g^2}\,\frac{m_p^2 c^2}{p^2}\,2^n n!\right)^{1/2n} \qquad (14)$$

or

$$r_0^{(n)} \sim \frac{\hbar}{\mu_\pi c}\left(\frac{\hbar\,100}{\mu_\pi c c\tau}\right)^{1/2n}\left(2^n n!\,\frac{\hbar c}{g^2}\right)^{1/2n}. \qquad (14')$$

For $n = 6$ we obtain for $r_0^{(6)}$ the estimate

$$r_0^{(6)} \sim 10^{-13}\ \text{cm}.$$

In other words, the quantities $r_0^{(n)}$ for the nucleon rapidly increase as the excitation number assumed to be characteristic of the nucleon increases, until for $n = 6$ the critical size, $r \sim 10^{-13}$ cm, is reached. These magnitudes depend only slightly on the size of the interaction constant $(g^2/\hbar c)^{1/2n}$.

The smallness of $(k_1, r_0^{(1)})$ in Eq. (10) also determines the smallness of the probability for the inverse process of the creation of a single particle for the interaction of a Λ^0 meson with a proton. It can be noted here that if the Λ^0 particle in the present model is associated with the first excited state of the nucleon, then single* creation of the Λ^0 particle is forbidden up to very high energies of the incident meson $(k^\pi \sim 1/r_0^{(1)})$. If we assume that the Λ^0 particle is represented by a nucleon of higher excitation, then $r_0^{(n)}$ becomes larger, and the energy region for the single creation of a Λ^0 particle moves toward smaller energies $(k^\pi \sim 1/r_0^{(n)})$. For $n = 5, 6$, $r_0^{(n)} \sim 10^{-13}$cm.

In other words, the creation of single Λ^0 particles from π^- mesons obtained from the cosmotron is not forbidden for energies of the π^- meson ~ 1.5 bev, $\bar{\lambda} \sim 2 \times 10^{-14}$cm for the case of a large n factor $(k^\pi r_0^{(n)})^{2n}$. If n were sufficiently large it would be possible to produce single Λ^0 particles by the scattering of such mesons on nucleons. We are concerned here with cross sections not much smaller than the cross sections for the scattering of mesons on nucleons.

The energy threshold for the creation of a single Λ^0 particle might be indicated within the framework of the present theory for the magnitude of r_0.

All these remarks apply also to those formulations which, while not giving a detailed mathematical description of the particle, explain its long lifetime by a large self-momentum. Unfortunately, correlation experiments, which in principle might give an answer to the question of the magnitude of the self-momentum of the Λ^0 particle are, as yet, given with poor statistics.

The indicated considerations are capable of explaining only some of the listed properties of the Λ^0 particles, that is, a) and b). Inasmuch as the Λ^0 particle is considered as an excited nucleon with all the resultant consequences from this for the interaction of the Λ^0 particle with the π^- meson field (one and the same $g^2/\hbar c$), then it is evident that the Λ^0 particle might be found in complex nuclei in a bound state (property c). Quantitative data concerning the energy binding of the Λ^0 particle in nuclei might here prove to be essential.

It is evident that properties d) and e) demand a new hypothesis, since here we encounter a new field (the Θ^0 field). In reference 4 an attempt is made to interpret these properties of the Λ^0 particles also, proceeding from the idea of excited nucleon states.

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* That is, without the accompaniment of Θ^0 particles: for example, $\pi^- + p \rightarrow \Lambda^0 + \pi^0$.

[4] M. A. Markov, Dokl. Akad. Nauk SSSR 101, 449 (1955)

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On the Problem of Anti-Protons in the Primary Stream of Cosmic Rays

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Analysis of data on the measurement of the east-west asymmetry of cosmic rays at high altitudes demonstrates that it is thoroughly impermissible to exclude the possibility of the presence of a certain proportion (up to 13%) of negatively charged particles in the primary flow. Computation shows that in the primary flow of cosmic rays, the proportion of anti-protons formed in interstellar space cannot exceed 0.2%, and apparently the presence of anti-protons cannot explain the results of measurement of east-west asymmetry.

COMPARISON of the intensity of cosmic rays measured at high altitudes arriving from the west with the intensity of cosmic rays arriving from the east, provides, as is generally known, the possibility of determining the sign of the charge on the particles in primary cosmic radiation. The magnitude of east-west asymmetry is characterized by the equation

$$A_{\mathrm{W-E}} = \frac{2(I_{\mathrm{W}} - I_{\mathrm{E}})}{I_{\mathrm{W}} + I_{\mathrm{E}}}, \qquad (1)$$

where I_{W}, I_{E} is the intensity of the cosmic rays arriving, respectively from the west and the east. If k is the percentage of negatively charged (negative) particles present in the primary radiation, then

$$A_{\mathrm{W-E}} = A^{+}_{\mathrm{W-E}}(1 - 0.02\,k), \qquad (2)$$

where $A^{+}_{\mathrm{W-E}}$ is the magnitude of east-west asymmetry when the cosmic rays consist solely of positively charged (positive) particles. Equation (2), which is valid if it is assumed that the energy spectra of particles of both signs are identical, makes it possible to find the proportion of negative particles in the primary stream of cosmic rays.

In 1949, Vernov, Grigorov, and others[1] carried out measurements of east-west asymmetry in the stratosphere and determined that for the hard component (passing through 8 cm of Pb) this magnitude comprises approximately 70%. This demonstrated that the dominant role in the primary stream of cosmic rays is played by positive particles. An analogous result was later obtained in other researches[2-6]. However, there is a definite divergence between measured east-west asymmetry and the asymmetry to be expected on the basis of purely positive primary articles. The integral energy spectrum of primary particles at definite energy intervals can be represented in the form of an exponential function $E^{-\gamma}$, where the exponent varies from $\gamma \approx 1$ in the range of energy near 1.4×10^{9} ev, to $\gamma \approx 2.5$ at energies greater 10^{13}ev. If the expected east-west asymmetry $A^{+}_{\mathrm{W-E}}$ is calculated and compared with the results of experiment[1] ($A_{\mathrm{W-E}} \approx 70\%$), then from Eq. (2) it is possible to find that proportion of negative particles which would make it possible to clarify the observed divergence. The magnitudes of $A^{+}_{\mathrm{W-E}}$ and k are given in the following table as calculated on the basis of various assumptions as to the size of the exponent γ.

γ	1	1.2	1.3	1.5
$A^{+}_{\mathrm{W-E}}$ in %	96	111	118	131
k in %	13.5	18.5	20	23

Even for a very "hard" ($\gamma = 1$) spectrum of primary particles, the results of measurement do not exclude the possibility of the presence of approximately 13% of negative particles in the primary cosmic ray stream. This result can vary somewhat, if the cascading of primary particles in the stratum of air above the instrument (order of magnitude of 15 g/cm^2) is taken into consideration. No correction for this cascading effect is required for measurements carried out beyond the boundaries of the atmosphere. Such measurements[5,6] showed $A_{W-E} \approx 40\% \pm 10\%$ while the expected figure was $A^{+}_{W-E} = 84\%$.. From this we obtain $k \approx 26$. Consideration of possible reverse flow of the particles ("albedo" of the cosmic rays) reduces the value of k to approximately 22,if the albedo (the ratio of the magnitude of reverse flow to the full magnitude of the measured flow) is.taken as equal to 15%, and can explain the entire divergence observed only if it is assumed that the albedo attains 52% at an angle of 45°, while direct measurements[7] permit a magnitude of the albedo not exceeding 10%.

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[1] S. N. Vernov, N. L. Grigorov, N. A. Dobrotin, S. P. Sokolov, F. D. Savin and A. I. Kurakin, Dokl. Akad. Nauk SSSR 68, 253 (1949)

[2] S. N. Vernov, A. M. Kulikov and A. N. Charakhch'ian Dokl. Akad. Nauk SSSR 85, 525 (1952)

[3] J. R. Winckler, T. Stix, K. Dwight and R. Sabin, Phys. Rev. 79, 656 (1950)

[4] B. Bhowmik, Phys. Rev. 89, 327 (1953)

[5] J. A. Van Allen and A. V. Gangnes, Phys. Rev. 79, 51 (1950)

[6] S. F. Singer, Phys. Rev. 80, 47 (1950)

Since direct experiments[8] point to the absence of any significant number of electrons of great energy in the primary cosmic ray stream, it is natural to assume that negative particles in the primary stream, if they are actually present there, are anti-protons. It is just such a conclusion in categorical form which is given by Bhowmik[4].

It is most natural to assume that anti-protons in cosmic rays (see Feinberg[9]) can form during the collision of nucleons of high energy. At the present time there is no conclusive theory dealing with the interaction of particles at high energies, but considerations developed by Fermi[10] and Landau[11] make it possible to carry out certain evaluations of the possible quantity of anti-protons in the primary stream of cosmic rays. Since we are interested in the upper limit for the number of anti-protons in the primary cosmic ray stream, we will make use of the results of the researches of Fermi, although these results, as can be seen from the remarks of Pomeranchuk[12] and Landau[11], definitely give inflated values for the number of nucleons and anti-nucleons which form.

It is reasonable to assume that the formation of anti-nucleons occurs solely in interstellar space and not in the primary sources of the cosmic rays, since the acceleration period for particles in the primary source is definitely less than the period between collisions leading to the appearance of anti-nucleons. For this reason the proportion of anti-protons *ap* in the primary stream of cosmic rays must be much less than the proportion of protons *p*, which are created in the main by the primary sources of the cosmic rays (for example, by super-novae[13]) or as a result of the breakdown of heavy nuclei.

Considering that the anti-protons in cosmic rays are in equilibrium with the protons, and ignoring the effect of ionization losses on the variation in the flow of anti-protons, we can write the balancing equations

$$\int_{E_{th}}^{\infty} N_p(E)\,\sigma_p(E)\,\mu(E)\,dE$$

$$= \int_{E_0}^{\infty} N_{ap}(E_1)\,[\sigma^{ap}_{coll} + \sigma_{ann}]\,dE_1$$

$$= -\int_{E_{th}}^{\infty} N_{ap}(E)\,\sigma^{ap}_{p}(E)\,\mu^{ap}(E)\,dE,$$

where $\mu(E)$ and $\mu^{ap}(E)$ are the average number of anti-protons with an energy greater than E_0, formed, respectively, from a proton and an anti-proton of energy E; $N_p(E)dE$ and $N_{ap}(E)dE$ are the differential spectra of the protons and anti-protons; $\sigma_p(E)$ and $\sigma^{ap}_p(E)$ are the effective cross-sections of the processes *p–p* and *ap – p* with the formation of nucleon pairs; σ^{ap}_{col} is the total cross-section of the process *ap – p*, and σ_{ann} is the cross-section of the annihilation of nucleonic pairs; $E_{th} \approx 10Mc^2$ is the minimum (threshold) energy at which the formation of nucleonic pairs is possible.

Making use of the equations taken from the work of Fermi, at $E < 500\,Mc^2$ we will have $\sigma_p < 0.02\sigma_{geom}$ and $\mu(E) \leq 0.75$, and at $E_{th} < E < 50\,Mc^2$, $\sigma_p < 0.01\sigma_{geom}$ and $\mu(E) \leq 0.5$. Naturally we must consider $\mu^{ap}(E) = \mu(E)$ and $\sigma^{ap}_p = \sigma_p$, and at $E > 1.8Mc^2$, $\sigma^{ap}_p = \sigma_p \approx \sigma_{geom}$. In conformity with reference 14, for the cross section of annihilation at $E > 10Mc^2$ we have $\sigma_{ann} < 0.002\sigma_{geom}$ and at

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[9] E. L. Feiberg, *Results of the Third Conference on Problems of Cosmogony*, p. 270, Publ. Acad. Sci., USSR, 1954

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[11] L. D. Landau, Izv. Akad. Nauk SSSR, Ser. Fiz. 17, 51 (1953)

[12] I. Ia. Pomeranchuk, Dokl. Akad. Nauk 78, 889 (1951)

[13] V. L. Ginzburg, Dokl. Akad. Nauk 92, 727 (1953)

[14] J. Ashkin, T. Auerbach and R. Marshak, Phys. Rev. 79, 226 (1950)

$E = 1.8Mc^2$, $\sigma_{ann} \approx 0.05\sigma_{geom}$.

In view of the fact that there are no particles with a momentum of less than $1.5Mc$ (see reference 15) in the primary stream of cosmic rays, we will now consider the quantity of anti-protons with $E > 1.8Mc^2$. Here, $E_0 = 1.8\ Mc^2$ and in Eq. (3) it is possible to ignore the cross section of annihilation in comparison with the cross section of collision.

The spectrum for the protons, in accordance with Neher[16], is written in the form

$$N_p(E)\,dE = N_0\,\varepsilon^{-3/5}\left[1 + 0.09\,\varepsilon^{4/5}\right]^{-3/5} d\varepsilon, \qquad (4)$$

where N_0 is a certain constant, $\epsilon = E/Mc^2$, M is the mass of the proton at rest.

The spectrum for the anti-protons is written, hypothetically, in the form of the exponential function

$$N_{ap}(E)\,dE = A\varepsilon^{-\xi}\,d\varepsilon.$$

Under the conditions given above, for $\xi = 2$, Eq. (3) is transformed into the inequality

$$N_{ap}(E > 1.8\ Mc^2) < 0.0027\ N_0\ Mc^2.$$

In the case of a "softer" spectrum ($\xi > 2$) the upper limit for the number of anti-protons in the primary stream of cosmic rays will be still lower. From Eq. (4) we find the entire number of protons with an energy of $E > 1.8\ MC^2$

$$N_p(E > 1{,}8\ Mc^2) \sim 1{,}6\ N_0\ Mc^2.$$

From this we find that the proportion of anti-protons in the primary stream of cosmic rays comprises

$$k < 0{,}0027\,/\,1.6 \approx 0{,}17\,\%.$$

For an energy $E > 10Mc^2$, which corresponds to measurements in the zone of the equator, we have

$$N_{ap}(E > 10\ Mc^2) < 0.0005\ N_0\ Mc^2,$$

$$N_p(E > 10\ Mc^2) \sim 1.4\ N_0\ Mc^2$$

and

$$k < 0{,}0005\,/\,1.4 \approx 0{,}04\,\%.$$

Although the factor k thus found is evidently excessive in value, even this factor does not conform with modern results of the measurement of east-west asymmetry ($k \sim$ 13-20%).

It is most probable, that in processing available experimental data with the aim of determining the proportion of negative particles, it is necessary to take more completely into consideration such effects as cascading in the atmosphere, reverse current, etc. Thus, the actual quantity of negative particles in the primary stream of cosmic rays is apparently small. However, in view of the importance of this problem, it appears expedient to carry out experiments with a view to the exact determination of the permissible proportion of these particles in primary cosmic rays. If it turns out that the proportion of anti-protons in the primary stream of cosmic rays comprises 0.1%, we may hope to set up special experiments that will isolate these particles

In conclusion I wish sincerely to thank Prof. V. L. Ginzburg, Prof. E. L. Feinberg, D. S. Chernavskii, and I. L. Rozental' for their evaluation of the results and their advice.

Translated by A. Certner
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Determination of Transverse Relaxation Times in the Magnetic Resonance of Atomic Nuclei in Weak High Frequency Magnetic Fields

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A method of measurement of the transverse relaxation times is developed, based on the determination of the interval of time between the extrema of the dispersion signal in a weak high-frequency magnetic field in the case of a nonadiabatic transition through resonance. Relations are obtained (and plotted graphically) which allow the determination of the time of transverse relaxation. As an experimental verification of the method, the dependence of the transverse relaxation time on the concentration of paramagnetic ions was measured in aqueous solutions of copper sulfate and ferric nitrate.

1. STATEMENT OF THE PROBLEM

UP to the present, the problem of the experimental determination of relaxation times in the magnetic resonance of atomic nuclei is still not completely clarified in the literature.

In the present work, a method is set forth for the determination of the times of transverse relaxation T_2 in a weak high-frequency magnetic field, i.e., when the following condition is satisfied[1]

$$\gamma^2 H_1^2 / a < 1, \tag{1}$$

where γ is the gyromagnetic ratio of the nucleus under consideration, H_1 is the half amplitude of the high-frequency magnetic field, a is a quantity proportional to the velocity modulation of the longitudinal magnetic field H_z.

$$a = |\gamma| \frac{dH_z}{dt}. \tag{2}$$

The theory of magnetic resonance for sufficiently small amplitude H_1 was given in references 1 and 2, where it was shown that in the case of linear modulation, when a = const, the dispersion of the signal $u(t)$ and the absorption $v(t)$ changed with time in the following manner:

$$u(t) = |\gamma| H_1 M_0 \times \int_{-\infty}^{t} e^{-(t-t')/T_2} \sin \frac{a}{2}(t^2 - t'^2)\, dt'; \tag{3}$$

$$v(t) = -|\gamma| H_1 M_0 \times \int_{-\infty}^{t} e^{-(t-t')/T_2} \cos \frac{a}{2}(t^2 - t'^2)\, dt', \tag{4}$$

where M_0 is the static magnetization.

If the process of transition through the resonance is carried out non-adiabatically, then the dispersion of the signal and the absorption have the form of curves with several extrema[2].

As has been shown[1,3], the signals u and v are related by the expression

$$\frac{du}{dt} + \frac{u}{T_2} + \Delta\omega(t)\, v = 0, \tag{5}$$

where

$$\Delta\omega(t) = at.$$

The proposed method of measuring T_2 is based on the determination of the interval of time between the first two extrema of the signal of dispersion u upon entering into the resonance region. At the instant when u has an extremum, $du/dt = 0$; therefore, for such times, we have, by Eq. (5),

$$aT_2 t = -u(t)/v(t). \tag{6}$$

If we now introduce the new variables $\eta = \sqrt{(a/2)\, t'}$ and denote

$$r = \sqrt{\frac{a}{2}}\, t, \quad \alpha = \frac{1}{T_2}\sqrt{\frac{2}{a}}, \tag{7}$$

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[1] S. D. Gvozdover and A. A. Magazanik, J. Exper. Theoret. Phys. USSR 20, 705 (1950)

[2] B. A. Jacobson and R. K. Wangsness, Phys. Rev. 73, 942 (1948)

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then Eq. (6) takes the form

$$\frac{2r}{\alpha} = \mathrm{tg}\,[r^2 - \psi(\alpha, r)], \tag{8}$$

where

$$\mathrm{tg}\,\psi(\alpha, r) = Q / P; \tag{9}$$

$$Q = \int_{-\infty}^{r} e^{\alpha\eta} \sin \eta^2 d\eta; \tag{10}$$

$$P = \int_{-\infty}^{r} e^{\alpha\eta} \cos \eta^2 d\eta; \tag{11}$$

$$Q = Q_0(\alpha) + Q_1(\alpha, r); \tag{12}$$

$$P = P_0(\alpha) + P_1(\alpha, r);$$

$$Q_0(\alpha) = \int_0^{\infty} e^{-\alpha\eta} \sin \eta^2 \, d\eta \tag{13}$$

$$= \sqrt{\frac{\pi}{2}} \left\{ \cos\frac{\alpha^2}{4}\left[0.5 - C\left(\frac{\alpha^2}{4}\right)\right] \right.$$

$$\left. + \sin\frac{\alpha^2}{4}\left[0.5 - S\left(\frac{\alpha^2}{4}\right)\right]\right\};$$

$$P_0(\alpha) = \int_0^{\infty} e^{-\alpha\eta} \cos \eta^2 d\eta$$

$$= \sqrt{\frac{\pi}{2}} \left\{ \cos\frac{\alpha^2}{4}\left[0.5 - S\left(\frac{\alpha^2}{4}\right)\right] \right.$$

$$\left. - \sin\frac{\alpha^2}{4}\left[0.5 - C\left(\frac{\alpha^2}{4}\right)\right]\right\};$$

S and C are Fresnel integrals;

$$Q_1(\alpha, r) = \int_0^{r} e^{\alpha\eta} \sin \eta^2 \, d\eta, \tag{14}$$

$$P_1(\alpha, r) = \int_0^{r} e^{\alpha\eta} \cos \eta^2 \, d\eta.$$

In this way the positions of the extrema of the signal u are determined by the transcendental Eq. (8), and for the determination of the time of transverse relaxation T_2 it is necessary to find the values of the roots of this equation.

2. DETERMINATION OF Q AND P FOR $r = \pm\alpha/2$

In the general case, the integrals Q and P [*Eqs.* (10) and (11)] are not expressed by tabulated functions. However, these integrals can be determined for two particular values $r = \pm\alpha/2$.

We multiply Eq. (10) by $j = \sqrt{-1}$ and add to it Eq. (11):

$$F = P + jQ = e^{j\alpha^2/4}\, W, \tag{15}$$

where

$$W = \int_{-\infty}^{r} \exp\left\{j\left(\eta - j\frac{\alpha}{2}\right)^2\right\} d\eta. \tag{16}$$

Noting that r is a function of α, we differentiate W with respect to α and integrate $dW/d\alpha$ for the conditions $r = \pm\alpha/2$, obtaining

$$[F]_{r=\pm\alpha/2} \tag{17}$$

$$= e^{j\alpha^2/4}\left[C_0 + {}^1/_2(-j \pm 1)\int_0^{\alpha} e^{\pm\eta^2/2} d\eta\right].$$

To determine the constant of integration C_0 in Eq. (17), we set $\alpha = 0$ and make use of Eq. (13):

$$[F]_{\substack{\alpha=0\\ r=0}} = \sqrt{\frac{\pi}{8}}(1 + j) = C_0.$$

Thus, we find from Eq. (17)

$$[P(\alpha, r)]_{r=-\alpha/2} \tag{18}$$

$$= \left[\cos\frac{\alpha^2}{4} - \sin\frac{\alpha^2}{4}\right]\sqrt{\frac{\pi}{8}}\left\{1 - \mathrm{erf}\left(\frac{\alpha}{\sqrt{2}}\right)\right\};$$

$$[Q(\alpha, r)]_{r=-\alpha/2}$$

$$= \left[\cos\frac{\alpha^2}{4} + \sin\frac{\alpha^2}{4}\right]\sqrt{\frac{\pi}{8}}\left\{1 - \mathrm{erf}\left(\frac{\alpha}{\sqrt{2}}\right)\right\},$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^{z} e^{-t^2} dt$$

is the error integral,

$$[P(\alpha, r)]_{r=\alpha/2} = \frac{1}{2}\left\{\left(\sin\frac{\alpha^2}{4} + \cos\frac{\alpha^2}{4}\right)\right. \tag{19}$$

$$\left. \times \int_0^{\alpha} e^{\eta^2/2} d\eta + \sqrt{\frac{\pi}{2}}\left(\cos\frac{\alpha^2}{4} - \sin\frac{\alpha^2}{4}\right)\right\},$$

$$[Q(\alpha, r)]_{r=\alpha/2} = \frac{1}{2}\left\{\left(\sin\frac{\alpha^2}{4} - \cos\frac{\alpha^2}{4}\right)\right.$$

$$\times \int_0^{\alpha} e^{\eta^2/2} d\eta + \sqrt{\frac{\pi}{2}}\left(\cos\frac{\alpha^2}{4} + \sin\frac{\alpha^2}{4}\right)\Bigg\}.$$

3. SEPARATION OF THE ROOTS OF THE TRANSCENDENTAL EQUATION (8) FOR $\alpha = 0$

To separate the roots of Eq. (8), we consider first the case $\alpha = 0$, i.e., by Eq. (7), $T_2 = \infty$. Rewriting Eq. (8) in the form

$$2r = \alpha \operatorname{tg}[r^2 - \psi(\alpha, r)],$$

we find that there is a minimum root, for which, from Eq. (9),

$$\text{for} \quad \alpha = 0, \quad r_{0,-1} = 0, \quad [\psi]_{\substack{\alpha=0\\ r=0}} = \pi/4. \tag{20}$$

Moreover, there exist a succession of roots for finite $r_{0,m}$ for which

$$\text{for} \quad \alpha = 0, \; r_{0,m}^2 - \psi(0, r_{0,m}) \tag{21}$$

$$= \frac{\pi}{2} + \pi(m-1), \quad m = 1,2,3,\ldots$$

The root of interest to us, in this case, nearest to $r_{0,-1}$, will correspond to $m = 1$ in view of Eq. (21).

$$r_{0.1}^2 - \psi(0, r_{0.1}) = \pi/2 \tag{22}$$

or, by Eqs. (9), (13), (14),

$$-\operatorname{ctg} r_{0.1}^2 = \operatorname{tg}\psi(0, r_{0.1}) \tag{23}$$

$$= \left[{}^1/_2\sqrt{\pi/2} + \int_0^{r_{0,1}} \sin\eta^2 d\eta\right]$$

$$\times \left[{}^1/_2\sqrt{\pi/2} + \int_0^{r_{0.1}} \cos\eta^2 d\eta\right]^{-1}$$

Representing Eq. (23) in the form

$$-\operatorname{tg}(r_{0.1}^2) = \frac{0.5 + C(r_{0.1}^2)}{0.5 + S(r_{0\,1}^2)},$$

and making use of the tables of Fresnel integrals[4], we find, by the method of iteration,

$$r_{0.1}^2 = 2.327, \tag{24}$$

whence

$$r_{0.1} = 1.525.$$

4. DETERMINATION OF THE FIRST ROOT OF r_{-1} IN THE GENERAL CASE

For the determination of the position of the first minimum root r_{-1} in the general case ($\alpha \neq 0$), we can, in conformity with Eq. (8), carry out the iteration according to the scheme

$$r_{-1}^{(n)} = \frac{\alpha}{2}\operatorname{tg}[(r_{-1}^{(n-1)})^2 - \psi(\alpha, r_{-1}^{(n-1)})], \tag{25}$$

where by $r_{-1}^{(n)}$ are meant the successive approximations of the desired root. We begin the iteration from the values determined by Eq. (20):

$$r_{-1}^{(1)} = \frac{\alpha}{2}\operatorname{tg}\left[0 - \frac{\pi}{4}\right] = -\frac{\alpha}{2}, \tag{26}$$

$$r_{-1}^{(2)} = \frac{\alpha}{2}\operatorname{tg}\left[\frac{\alpha^2}{4} - (\psi)_{r=-\alpha/2}\right]. \tag{27}$$

But, by Eqs. (9) and (18),

$$[\operatorname{tg}\psi]_{\substack{\alpha\\ r=-\alpha/2}} = \operatorname{tg}\left[\frac{\pi}{4} + \frac{\alpha^2}{4}\right],$$

i.e.,

$$[\psi]_{r=-\alpha/2} = \frac{\pi}{4} + \frac{\alpha^2}{4},$$

whence, in accord with Eq. (27),

$$r_{-1}^{(2)} = -\alpha/2. \tag{28}$$

Inasmuch as the two successive approximations (26) and (28) coincide, the exact value of the first extremum is

$$r_{-1} = -\alpha/2. \tag{29}$$

Thus, the first extremum of the signal sets in for negative values of r, i.e., up to the onset of resonance, which corresponds to $r = 0$.

5. DETERMINATION OF THE POSITION OF THE SUBSEQUENT ROOT IN THE GENERAL CASE

To determine the position of the subsequent root in Eq. (8) for the case $\alpha \neq 0$, we must know the values of the integrals Q and P [Eqs. (10) and (11)]. We introduce the quantity

$$I = P_1(\alpha, r) + jQ_1(\alpha, r) \tag{30}$$

$$= e^{j\alpha^2/4}\int_0^r \exp\left\{j\left(\eta - j\frac{\alpha}{2}\right)^2\right\} d\eta.$$

and the new variable

$$t = \sqrt{j}\left(\eta - j\frac{\alpha}{2}\right) = x + jy, \tag{31}$$

where

$$x = \left(\eta + \frac{\alpha}{2}\right)/\sqrt{2}, \quad y = \left(\eta - \frac{\alpha}{2}\right)/\sqrt{2},$$

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[4] E. Jahnke and F. Emde, *Tables of Functions*

we get

$$I = \exp\left\{j\left(\frac{\alpha^2}{4} - \frac{\pi}{4}\right)\right\}\int_{t_1}^{t_2} e^{t^2}dt. \qquad (32)$$

The limits of integration have the following values:

$$\text{for} \quad \eta = 0, \quad x_1 = \alpha/2\sqrt{2}, \qquad (33)$$

$$y_1 = -\alpha/2\sqrt{2};$$

$$\text{for} \quad \eta = r, \quad x_2 = \left(r + \frac{\alpha}{2}\right)\Big/\sqrt{2},$$

$$y_2 = \left(r - \frac{\alpha}{2}\right)\Big/\sqrt{2}.$$

We carry out the integration over two intervals of the complex variable:

$$\int_{t_1}^{t_2} e^{t^2}dt = \int_0^{t_2} e^{t^2}dt - \int_0^{t_1} e^{t^2}dt. \qquad (34)$$

From Eq. (34), we see that in the integration from zero to t_1,

$$t = (1 - j)x.$$

Denoting

$$2x^2 = \zeta^2,$$

we get

$$\int_0^{t_1} e^{t^2}dt = e^{-j(\pi/4)}\int_0^{\alpha/2} e^{-j\zeta^2}\,d\zeta. \qquad (35)$$

Upon substitution of Eqs. (34) and (35) in Eq. (32), I takes the form

$$I = \exp\left\{j\left[\frac{\alpha^2}{4} - \frac{\pi}{4}\right]\right\}\int_0^{t_2} e^{t^2}dt + je^{j\alpha^2/4} \qquad (36)$$

$$\times \int_0^{\alpha/2} e^{-j\zeta^2}d\zeta = I_2 + I_1,$$

where I_2 and I_1 denote the first and second terms, respectively. The integral which enters into I_1 is calculated with the aid of the Fresnel integrals

$$\int_0^{\alpha/2} e^{-j\zeta^2}d\zeta = \sqrt{\frac{\pi}{2}}\left\{C\left(\frac{\alpha^2}{4}\right) - jS\left(\frac{\alpha^2}{4}\right)\right\}.$$

The real and imaginary parts of I_1 have the form

$$\operatorname{Re} I_1 = \sqrt{\frac{\pi}{2}} \qquad (37)$$

$$\times \left[S\left(\frac{\alpha^2}{4}\right)\cos\frac{\alpha^2}{4} - C\left(\frac{\alpha^2}{4}\right)\sin\frac{\alpha^2}{4}\right],$$

$$\operatorname{Im} I_1 = \sqrt{\frac{\pi}{2}}$$

$$\times \left[S\left(\frac{\alpha^2}{4}\right)\sin\frac{\alpha^2}{4} + C\left(\frac{\alpha^2}{4}\right)\cos\frac{\alpha^2}{4}\right].$$

The integral entering into I_2 is calculated with the aid of tables of the real and imaginary parts of the probability integral of complex argument[5]

$$\int_0^{t_2} e^{t^2}dt = \frac{\sqrt{\pi}}{2}\{e^{t_2^2}[V(x_2, y_2) \qquad (38)$$

$$- jU(x_2, y_2)] + j\},$$

where U and V denote, respectively, the real and imaginary parts of the probability integral.

Expressing x_2 and y_2 in terms of α and r and substituting Eq. (38) in Eq. (36), we find the expression for I_2

$$I_2 = \frac{\sqrt{\pi}}{2}\left[e^{\alpha r}\exp\left\{j\left(r^2 - \frac{\pi}{4}\right)\right\}(V - jU) \qquad (39)\right.$$

$$\left. + j\exp\left\{j\left(\frac{\alpha^2}{4} - \frac{\pi}{4}\right)\right\}\right].$$

Separating the real and imaginary parts, we have

$$\operatorname{Re} I_2 = \frac{\sqrt{\pi}}{2\sqrt{2}}\left\{e^{\alpha r}[(V - U)\cos r^2 \qquad (40)\right.$$

$$\left. + (V + U)\sin r^2] + \left[\cos\frac{\alpha^2}{4} - \sin\frac{\alpha^2}{4}\right]\right\};$$

$$\operatorname{Im} I_2 = \frac{\sqrt{\pi}}{2\sqrt{2}}\left\{e^{\alpha r}[(V - U)\sin r^2\right.$$

$$\left. (V + U)\cos r^2] + \left[\cos\frac{\alpha^2}{4} + \sin\frac{\alpha^2}{4}\right]\right\}.$$

Making use of Eqs. (12), (13), (30), (37) and (40), we find the final expressions for the integrals Q and P:

$$Q = \int_{-\infty}^{r} e^{\alpha\eta}\sin\eta^2\,d\eta \qquad (41)$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{e^{\alpha r}[(V - U)\sin r^2 - (V + U)\cos r^2]\right.$$

$$\left. + 2\left[\cos\frac{\alpha^2}{4} + \sin\frac{\alpha^2}{4}\right]\right\};$$

$$P = \int_{-\infty}^{r} e^{\alpha\eta}\cos\eta^2\,d\eta \qquad (42)$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{e^{\alpha r}[(V - U)\cos r^2 + (V + U)\sin r^2]\right.$$

$$\left. + 2\left[\cos\frac{\alpha^2}{4} - \sin\frac{\alpha^2}{4}\right]\right\}.$$

Substituting Eqs. (41) and (42) into Eq. (8), we get the equation for the determination of the positions of the extrema of the function $u(t)$:

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[5] V. N. Faddeeva and N. M. Terent'ev, *Tables of the Values of the Probability Integral of Complex Argument*, GITTL, 1954

$$r = \frac{\alpha}{2}\,\frac{V + U + 2e^{-\alpha r}\left[\sin\left(r^2 - \frac{\alpha^2}{4}\right) - \cos\left(r^2 - \frac{\alpha^2}{4}\right)\right]}{V - U + 2e^{-\alpha r}\left[\sin\left(r^2 - \frac{\alpha^2}{4}\right) + \cos\left(r^2 - \frac{\alpha^2}{4}\right)\right]}\,; \tag{43}$$

$\alpha/2$	r_{+1}	Δr	$\alpha/2$	r_{+1}	Δr
0.0000	1.525	1.525	1.4142	1.63	3.04
0.1414	1.49	1.63	1.7680	1.78	3.55
0.3535	1.46	1.81	2.4748	2.16	4.63
0.7071	1.45	2.16			

where U and V are functions of x_2 and y_2, determined in accordance with Eq. (33).

In order to find the positions of the subsequent extremum r_{+1}, Eq. (43) is solved by interpolation by the chord method. The results which were computed appear in the Table and are drawn in Fig. 1 in the form of the curve r_{+1}. The curve $\Delta r = r_{+1} - r_{-1} = r_{+1} + \alpha/2$ appears in this same drawing.

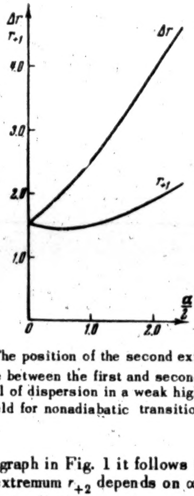

FIG. 1. The position of the second extremum r_{+1} and the distance between the first and second extrema Δr of the signal of dispersion in a weak high-frequency magnetic field for nonadiabatic transition through resonance.

From the graph in Fig. 1 it follows that the position of the extremum r_{+2} depends on $\alpha/2$, and for an increase of $\alpha/2$ from a value ≈ 0.6, the quantity r_{+1} increases monotonically, remaining, however, always less than $\alpha/2$.

6. THE VALUE OF THE ROOT r_{+1} AS $\alpha/2 \to \infty$

For a decrease in the transverse relaxation time T_2 (i.e., as $\alpha \to \infty$) the process of transition through resonance becomes adiabatic, as is known[3], and corresponds to the case of "slow passage". In this case, the signal of dispersion has only two extrema, located at equal distances from the moment of resonance. It can be shown that for $\alpha \to \infty$ the value $r_{+1} \to \alpha/2$. We make use of the iteration scheme (25), given in Sec. 4, and assume that $r_{+1}^{(1)} = \alpha/2$. Then,

$$r_{+1}^{(2)} = \frac{\alpha}{2}\,\mathrm{tg}\left[\frac{\alpha^2}{4} - (\psi)_{r=\alpha/2}\right]. \tag{44}$$

From Eqs. (9) and (19), it follows that as $\alpha \to \infty$,

$$[\mathrm{tg}\,\psi]_{\substack{r=\alpha/2\\ \alpha\to\infty}} = \mathrm{tg}\left(\frac{\alpha^2}{4} - \frac{\pi}{4}\right),$$

i.e.,

$$[\psi]_{\substack{r=\alpha/2\\ \alpha\to\infty}} = \frac{\alpha^2}{4} - \frac{\pi}{4},$$

while from Eq. (44)

$$r_{+1}^{(2)} = \alpha/2. \tag{45}$$

Since, as $r = \alpha/2, \alpha \to \infty$, Eq. (44) becomes an indentity, the position of the following extremum becomes

$$(r_{+1})_{\alpha\to\infty} = \alpha/2. \tag{46}$$

and, consequently,

$$r_{+1} = |r_{-1}|.$$

7. THE CHANGE OF THE FORM OF THE DISPERSION OF THE SIGNAL

According to theory[1,2], the form of the observed signals depends on the magnitude of the parameter $\sqrt{a}\,T_2$. For values of $\sqrt{a}\,T_2 \geq 1$, the vibrations appear beyond resonance. Keeping in mind that, by Eq. (7)

$$\sqrt{a}\,T_2 = \sqrt{2}/\alpha,$$

and taking into consideration the graphs in Fig. 1, we find that the form of the dispersion of the signal must change in the following way; upon a decrease in $\alpha/2$, the distance between the first and second extrema (Δr) must decrease and, beginning with $\alpha/2 < 1$, vibrations must appear whose amplitude increases upon further decrease in $\alpha/2$.

Oscillograms of the dispersion of the signal are given in Fig. 2. These illustrate the change in the shape of the signals for change in $\alpha/2$. The oscillograms were obtained for aqueous solutions of different concentrations of copper sulfate and ferric nitrate. All the oscillograms were taken under identical conditions. The corresponding values of $\alpha/2$ are given in the captions for the figures. The oscillograms verify the conclusions of the theory of the dependence of the form of the curves on $\alpha/2$.

8. METHOD OF DETERMINATION OF T_2

In the practical application of this method, the dispersion of the signal is observed in a weak high-frequency magnetic field of frequency ω[6]. To the constant magnetic field H_0, there is added a magnetic field of amplitude H_m and audio frequency ω_m, such that the resultant magnetic field is equal to

$$H_z = H_0 + H_m \sin \omega_m t. \tag{47}$$

If the signal $u(t)$ is located in a linear segment at the center of the oscillogram, then, by Eq. (47),

$$a = |\gamma| H_m \omega_m. \tag{48}$$

Behind the linear part of the modulation there are regions where the velocity of modulation and the time of transition through resonance are constant within the limits of accuracy chosen. For the determination of T_2, it is necessary that the distance between the first and second extrema lie in the linear part, while the remaining part of the signal $u(t)$ can be located in intervals with varying velocity; it is only important that the signal be damped out at the onset of the following resonance, which arises on the reverse path of the development.

Measuring (along the oscillogram) the distance between the first and second extrema and knowing the amplitude and frequency of the modulating field, we can determine the time t. Then, by Eqs. (7) and (48),

$$\Delta r = \sqrt{\frac{a}{2}}\,\Delta t = \sqrt{\frac{|\gamma| H_m \omega_m}{2}}\,\Delta t. \tag{49}$$

We determine $\alpha/2$ graphically for a given Δr from Fig. 1, and compute T_2 from Eq. (7):

$$T_2 = \frac{1}{\sqrt{2a}}\left(\frac{2}{\alpha}\right) = \frac{1}{\sqrt{2|\gamma| H_m \omega_m}}\left(\frac{2}{\alpha}\right). \tag{50}$$

The error in the determination of T_2 arising from the use of Eq. (50), is determined by the errors in the measurement of H_m, ω_m, Δt and the error in the determination of $\alpha/2$ from the graph.

If the above calculated quantities are determined with possible errors H_m: 1%, ω_m: 1%, Δt: 3%, $\alpha/2$: 1%, then for the case that Δr lies at the middle of the graph, i.e., for $2 < \Delta r < 4$, (which ordinarily takes place for suitable choice of the velocity of modulation), the maximum possible error in determining T_2 amounts to 8-15%. For smaller values of Δr, the error in the determination of T_2 increases.

The present method of determining T_2 is sufficiently simple and gives an accuracy not less than the methods described earlier in the literature. The principal advantage of this method is the fact that one can work in a nonstabilized constant magnetic field, which greatly simplifies the arrangement for the measurement of the time of transverse relaxation.

9. EXPERIMENTAL TEST OF THE METHOD

To test the method just described, the values of the transverse relaxation times T_2 were measured for aqueous solutions of different concentrations of copper sulfate ($CuSO_4$) and ferric nitrate [$Fe(NO_3)_3$]. The dependence of T_2 on the concentration of the paramagnetic ions Cu^{++} (curves *1* and *2*) and of the ions Fe^{+++} (curve *3*) in the solution is indicated in Fig. 3. The accuracy of measurement of T_2 was about 10%.

It follows from Fig. 3 that for reduced concentrations the transverse relaxation time approaches a limit whose magnitude is determined by the inhomogeneous constant magnetic field. The amount of inhomogeneity of the constant magnetic field along the specimen ΔH_0 was estimated by the value of T_2, measured for distilled water by the formula

$$\frac{1}{T_2} = \gamma(H_{\text{int}} + \Delta H_0) = \frac{1}{T_{2\,\text{вн}}} + \frac{1}{T_{2\,\text{int}}}, \tag{51}$$

where H_{int} = intensity of the internal field, $T_{2\,\text{int}}$

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[6] S. D. Gvozdover and N. M. Ievskaia, J. Exper. Theoret. Phys. USSR 25, 435 (1953)

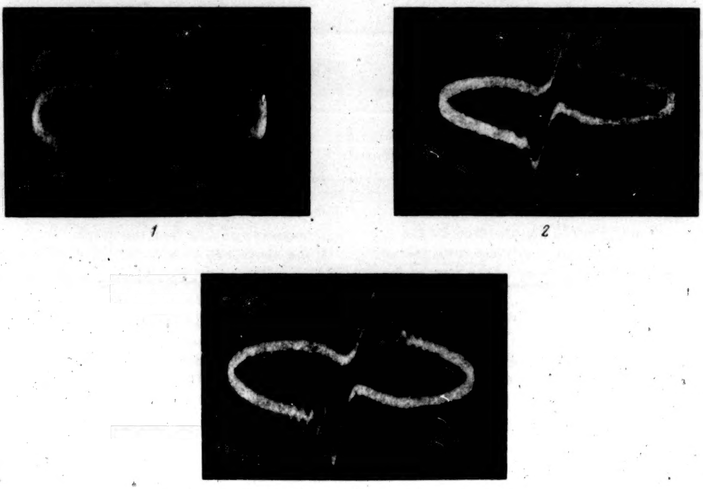

FIG. 2. The change in shape of the signals of dispersion in dependence on $\alpha/2$. The oscillograms were obtained (1) from $0.5M$ solution of $Fe(NO_3)_3$; (2) from $1M$ solution $CuSO_4$; (3) from $0.2M$ solution of $CuSO_4$ under the following conditions: frequency of modulation 50 cps, amplitude of modulation 2.38 oersteds, half amplitude of high-frequency field, 0.03 oersted. Values of $\alpha/2$: 1 – 1.35; 2 – 0.85; 3 – 0.46.

and T_{inhom} are the values of the transverse relaxation times, determined respectively for the internal field and the inhomgeneity. (As is known[7], for distilled water, $T_{2\,\text{int}} \gg T_{2\,\text{inhom}}$.)

In the results shown in Fig. 3, (curves 4, 5, 6) the value of the inhomogeneity was respectively equal to 0.072; 0.081; 0.11 oersted, which corresponds to 2.43×10^{-3}; 2.72×10^{-3}; 3.72×10^{-3}% of the value of the constant field.

The dependence of $T_{2\,\text{int}}$, determined according to Eq. (51) by taking into account the effect of the inhomogeneity, on the concentration of the paramagnetic ions is shown in Fig. 3 for Cu^{++} ions (curve 7) and for the Fe^{+++} ions (curve 8). It follows from the drawing that the dependence of $T_{2\,\text{int}}$ on the concentration of the paramagnetic ions is a straight line on the logarithmic scale. The lines for the aqueous solutions of copper sulfate and ferric nitrate are parallel to each other and are inclined at 45° with respect to the abscissa. Such a dependence shows that the transverse relaxation time $T_{2\,\text{int}}$ produced by the internal field is inversely proportional to the concentration of the paramagnetic ions.

A comparison of the results of measurements with the data given by Bloembergen[7] (curve 9) shows that the value of T_2 for aqueous solutions of ferric nitrate, obtained in the present research for large concentrations, where the inhomogeneity is small, agrees with the results of Bloembergen within the limits of experimental error. For small concentrations, the resultant values of $T_{2\,\text{int}}$ differ from the T_2 obtained in reference 7 by about

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[7] N. Bloembergen, *Nuclear Magnetic Relaxation*, The Hague, 1948

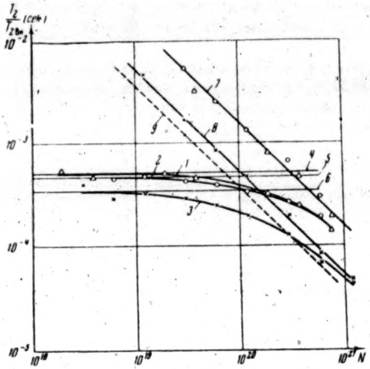

FIG. 3. Dependence of T_2 and $T_{2\,\mathrm{int}}$ on the concentration of paramagnetic ions N for aqueous solutions of $CuSO_4$ and $Fe(NO_3)_3$. The concentration N is expressed in number of ions/cm^3. Measurements for Cu^{++} ions: Δ for conditions when $T_{2\,\mathrm{inhom}}$ is determined by curve *4*; O for conditions when $T_{2\,\mathrm{inhom}}$ is determined by curve *5*; measurements for Fe^{+++} ions under conditions when $T_{2\,\mathrm{inhom}}$ is determined by curve *6*.

40%. This difference can be explained by the fact that the accuracy of the determination of $T_{2\,\mathrm{int}}$ is significantly less as a result of errors in the measurement of $T_{2\,\mathrm{inhom}}$.

Thus, the agreement of the results within the limits of experimental error confirms the possibility of application of the method developed here.

Translated by R. T. Beyer
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The Possibility of the Formation of Λ^0-Particles by Protons with Energies up to 700 mev*

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An attempt was made to observe the formation of Λ^0-particles in collisions of 670 mev protons with carbon nuclei. In principle, the experiment allowed the detection of Λ^0-particles with the decay scheme $\Lambda^0 \to n + \pi^0$. Gamma-rays from the decay of π^0-mesons were detected by means of a telescope of scintillation and Cerenkov counters. It was found that the cross section for the creation of Λ^0-particles has a value of $\sigma_{\Lambda^0} \lesssim 10^{-31}$ cm^2/nucleon. Conclusions are drawn on the mechanism of creation of Λ-particles.

INTRODUCTION

THE mechanism of the creation of hyperons is unknown as yet. In earlier papers[2,3], difficulties were pointed out as to the explanation of the existence of heavy mesons and hyperons (in particular, the Λ^0-particles) which are created with a high probability in high energy collisions and have long lifetimes. An attempt to eliminate such difficulties leads to the following alternative: either the process of the formation of Λ^0-particles is not the reverse process of their decay, or, conversely, the process of their creation is the reverse of the decay process; but then the matrix element depends exceptionally strongly on the energy. This strong dependence of the matrix element on the energy could be found, for instance, in the case of a very large spin of the particles.

The first possibility was examined in references 2 and 3, where a conclusion was reached on the basis of a phenomenological analysis of the experimental data relative to the creation of Λ^0-particles and heavy mesons as to the possibility of the following transformation scheme of the nucleon:

$$(N) \rightleftarrows (\Lambda^0) + (\text{heavy meson}) \; (N \equiv \text{nucleon}) \qquad (1)$$

The above scheme describes a virtual transformation of the nucleon, analogous to the Yukawa process $(N) \rightleftarrows (N) + (\pi)$. The heavy meson requires a strong interaction between the nucleon and the Λ^0-particle. The scheme (1) permits us to put aside the difficulties connected with the duration of the lifetime and the probability of formation, not only of the Λ^0-particles, but also of the heavy mesons.

Certain indications of the correctness of the scheme (1) were obtained, for example, in the experiments of Fowler et al[4], performed with the cosmotron. In these experiments, simultaneous formation of a Λ^0-particle and a heavy neutral meson was observed upon bombarding hydrogen with π^--mesons with an energy of 1500 mev.

THE EFFECTIVE THRESHOLD FOR THE FORMATION OF Λ^0-PARTICLES

Let us first examine the reactions of formation of the Λ^0-particles in nucleon-nucleon collisions. We will assume the above-described scheme (1) as correct. If that is the case, the cross section for the $N + N \to \Lambda^0 + N$ reaction, which has a threshold of 371 mev*, should be exceedingly small. Among the processes which in this case could have a signigicant cross section, we will examine the following:

$$N + N \to N + \Lambda^0 + \text{heavy meson} \qquad (2)$$

$$N + N \to \Lambda^0 + \Lambda^0 \qquad (3)$$

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[1] M. P. Balandin, B. D. Balashov, V. A. Zhukov, B. M. Pontecorvo and G. I. Selivanov, Report of the Inst. for Nuclear Problems, Acad. Sci., USSR (1954)

[2] B. M. Pontecorvo, Report of the Inst. for Nuclear Problems, Acad. Sci., USSR (1951); J. Exper. Theoret. Phys. USSR 29, 140 (1955); Soviet Phys. 2, 135 (1956)

[3] A. Pais, Phys. Rev. 86, 663 (1952)

* The present paper is based on the results of work performed in 1954 and was described earlier in a report of the Institute for Nuclear Problems of the Academy of Sciences of the USSR[1].

* The mass of the Λ^0-particle is assumed to be equal to 2182 m_e.

[4] W. F. Fowler, R. P. Shutt, A. M. Thorndike and W. L. Whittemore, Phys. Rev. 93, 861 (1954)

The effective threshold for the formation of Λ^0-particles will be understood as being the lowest value among the possible thresholds corresponding to the reactions (2) and (3). The threshold of (2) cannot be evaluated because the mass of the heavy meson, M_T, which participates in the reaction, is unknown. However, if it is assumed that $M_T = 965\ m_e$, the threshold of the reaction (2) becomes 1577 mev. The reaction (3), according to the scheme (1), should proceed via intermediate virtual transitions which represent, in the first place, the emission of a Λ^0-particle simultaneously with that of a heavy meson, and then the absorption of this meson followed by the formation of a second Λ^0-particle. The threshold for this reaction is 775 mev.

In the case of the collision of a nucleon with a complex nucleus, the effective threshold δ becomes significantly lower and can be evaluated in the momentum approximation, if the intranuclear motion of the nucleons is taken into account. Assuming a maximum "Fermi energy" of the nucleons inside of the nucleus equal to 20 mev, we obtain the value $\delta \simeq 450$ mev. It is interesting to observe that in the estimate of the threshold for the formation of the Λ^0-particles by nucleons on nuclei, no difficulties arise from the Pauli principle, such difficulties being found in the evaluation of the threshold for the formation of π-mesons by nucleons on nuclei.

Passing on to the processes of formation of Λ^0-particles by π-mesons we will consider the reaction $\pi + (N, N) \rightarrow \Lambda^0 + N$. Its threshold is equal to 40 mev. As in the previous case, with our former assumption, the cross section of this reaction should be extremely small. In connection with this, it is advisable to examine the following two reactions:

$$\pi + N \rightarrow \Lambda^0 + \text{heavy meson} \tag{4}$$

$$\pi + (N, N) \rightarrow \Lambda^0 + \Lambda^0. \tag{5}$$

If $M_T = 965\ m_e$, the threshold of (4) will be 757 mev. The threshold of the reaction (5) has the value of 242 mev.

A SHORT SURVEY OF THE PAPERS RELATED TO THE FORMATION OF HYPERONS BY PARTICLES WITH AN ENERGY UP TO 500 MEV

Bernardini and Segre[5] established the fact that the cross section for the formation of Λ^0-particles by bremsstrahlung photons of a maximum energy of 330 mev constitute 10^{-4} of the cross section for the creation of charged π-mesons.

The process of the formation of Λ^0-particles in collisions of protons with carbon nuclei was investigated in references 6 and 7. The cross section for this process, as found in reference 6, for a proton energy of 430 mev, has a value of less than $0.35 \times 10^{-27}\ \text{cm}^2$. According to more refined experiments performed by Garwin[7], the value of the cross section for the formation of Λ^0-particles in p-C collisions, even for a proton energy of 450 mev, does not exceed $10^{-31}\ \text{cm}^2$.

A paper by Schein et al[8] is devoted to the formation of Λ^0-particles by π^--mesons on carbon, at an energy of 227 mev. In this work, 5 cases of the creation of Λ^0-particles were observed on photographic plates, but the value of the corresponding cross section is not quoted. Nevertheless, there is reason to believe that the preliminary data published by Schein contradict the results of Garwin's work. Therefore, it seemed very interesting to try to observe the formation of Λ^0-particles by nucleons of energies between 500 and 1000 mev.

EXPERIMENTAL ARRANGEMENT

The basis of the present experiment is the method proposed by Garwin. In principle, this method allows us to observe Λ^0-particles which disintegrate according to the scheme

$$\Lambda^0 \rightarrow n + \pi^0 \tag{6}$$

The idea is that Λ^0-particles can be identified by means of the π^0-mesons being created at a certain distance from the target bombarded by protons.

Notwithstanding the fact that the decay of Λ^0-particles with the emission of π^0-mesons has not been observed, we may affirm with certainty that the probability of the process (6) must be comparable with the probability of decay according to the scheme $\Lambda^0 \rightarrow p + \pi^-$. Such an affirmation is based on the experimental fact that the process of exchange scattering of π-mesons by nucleons (the

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[5] G. Bernardini and A. Segre, Quoted in Phys. Rev. 92, 727 (1953)

[6] A. H. Rosenfeld and S. B. Treiman, Phys. Rev. 92, 727 (1953)

[7] R. L. Garwin, Phys. Rev. 90, 274 (1953)

[8] M. Schein, D. Haskin, R. Glasser, F. Fainberg and K. Brown, Congres International sur le rayonnement cosmique, Bagnere de Bigorre (1953)

$\pi^- + p \rightarrow \pi^0 + n$ reaction) has a high probability.

The scheme of the experiment is indicated in Fig. 1.

The graphite target was irradiated by protons with an energy of 670 mev, which were accelerated in the synchrocyclotron of the Institute for Nuclear Problems. The displacement of the target, radially and azimuthally, was performed by remote control. The remote control provided a significant reduction of the working time for the experiment and in addition permitted measurements under steady operating conditions of the synchrocyclotron. With the aid of an electrical indicating system provided with a servo potentiometer, the position of the target was indicated with an accuracy of ±0.025 cm. The direction of motion of the circulating beam of protons was changed by reversing the magnetic field, H, of the synchrocyclotron (in Fig. 2 this is shown by the corresponding arrows). The collimator of γ-rays was directed along the radius of the equilibrium orbit of the circulating beam. It consisted of 3 blocks of lead, all being [illegible] cm long. All blocks had rectangular openings, 0.7 × 1.5 cm. One of these blocks was placed in the 60 × 60 × 60 cm lead shield, at a distance of 210 cm from the target. The two other blocks were placed in the iron shield, 2.7 m thick. The distance between the extreme blocks was 4 m. The γ-ray detector was placed at a distance of 8.5 m from the target. Around the detector a 15 cm thick iron shield was placed; in front, there was an additional lead shield, 60 cm thick.

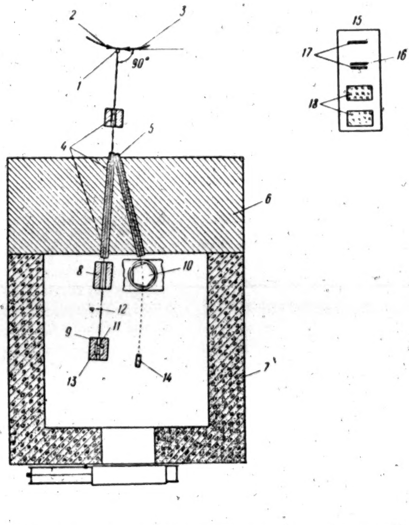

FIG. 1. Schematic diagram of the experiment. *1.* remote controlled target, 0.25 cm of C; *2.* beam of protons at reversed H; *3.* beam of protons for normal H; *4.* γ-ray collimator; *5.* π-meson collimator; *6,9.* iron shielding; *7.* concrete shielding; *8.* supplementary lead shielding; *10.* deflecting magnet; *11,16.* converter *1*; *12.* converter *2*; *13.* γ-ray detector; *14.* π-meson detector; *15.* γ-ray detector in an amplified view; *17.* toluene scintillation counter (upper, A; lower, B); *18.* Cerenkov counters, of plexiglass (upper, C; lower, D).

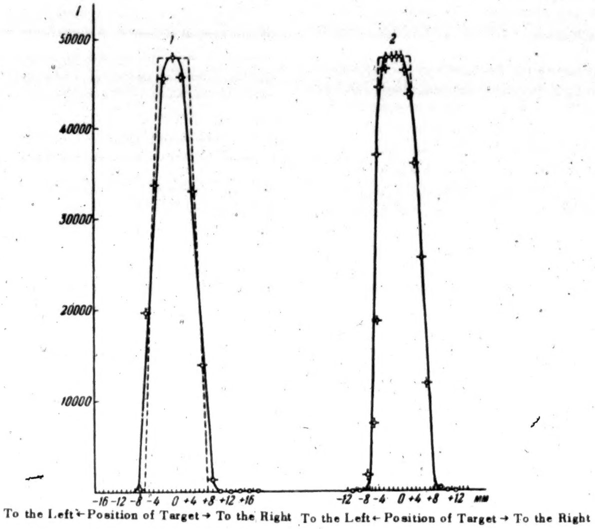

FIG. 2. Dependence of the intensity of γ-rays at the outlet of the collimator (an 0.7 × 1.5 cm opening) on the azimuth of the target. The full curve *1* was calculated for a target thickness of 0.5 cm; the full curve *2*, for a thickness of 0.25 cm. The dotted curves correspond to a point target.

THE γ-RAY DETECTOR

The γ-ray detector consisted of a telescope of two scintillation counters (*A*, *B*) and two Cerenkov counters (*C*, *D*). Toluene crystals were used as scintillators. The crystal of the counter *A* had the dimensions 5 × 5 × 0.5 cm, while *B* was 4 × 4 × 0.5 cm. In the Cerenkov counters, plexiglass of cylindrical form was used, with a diameter of 6 cm and a thickness of 3.5 cm. The light pulses from the toluene and plexiglass were registered by photomultipliers of the FEU-19 type. The pulses from each counter went through an amplifier with a pass band of 50 mc and an amplification coefficient of 100 and were fed into the input of a triple coincidence circuit. The counter *A* was connected for anticoincidence.

Events of the BCD-A type were registered, which corresponded to the simultaneous appearance of pulses in the three counters (*B*, *C*, *D*). The telescope was provided with two lead converters (*1*, *2*), both 4 × 4 × 0.5 cm. The arrangement of the converters is shown in Fig. 1. The intensity of the γ-rays, *I*, was determined by measuring the difference in the number of events $(BCD\text{-}A)_1$ given by the arrangement with the converter *1*, but without the converter *2*, and the number of events $(BCD\text{-}A)_2$ with the converter *2* and without *1*. At the position of the target which coincided with the axis of the collimator, the frequency of the events $(BCD\text{-}A)_2$ was 2.5-3% of the frequency of the events $(BCD\text{-}A)_1$.

THE MONITOR

As a monitor, a BF_3-filled counter was used, being placed out of the limits of the lead shield indicated in Fig. 1. To clarify the possibility of using the BF_3-counter as a monitor, the following experiments were performed:

1) The indications of the BF_3-counter and those

of a supplementary telescope of three scintillation counters were registered simultaneously. With this telescope the π^--mesons passing a separate collimator and a deflecting magnet were registered (Fig. 1). On changing the intensity of the synchrocyclotron by a factor of several times,the indication of the BF_3 counter changed proportionally to the indication of the π^--meson detector.

2) The indications of the BF_3 counter were registered at various azimuthal positions of the target. The use of remote control permitted the rapid return of the target to its original position, without interruption of the operation of the synchrocyclotron. We could thus estimate the constancy of the proton beam intensity during the measurements. It was found that at a constant intensity of the circulating proton beam the indications of the counter do not depend on the displacements of the target of several centimeters.

Thus, the control experiments performed showed that the BF_3 counter can be used as a satisfactory monitor in our experiment.

MEASUREMENTS

The initial measurements constituted an investigation of the properties of the collimator. With a graphite target 0.5 cm thick, the intensity of γ-rays, I, was plotted as a function of the azimuthal target position. The obtained experimental data are represented in Fig. 2. The errors indicated in this figure are statistical. In the same graph the errors introduced by the inaccuracy in the determination of the position of the azimuth of the target are also given.

According to the experimental data obtained with a target of the originally specified thickness, with this thickness it was not possible to determine the dimensions of the region of visibility of the target. For this reason, a thinner target of graphite was used, 0.25 cm thick, during the fundamental measurements. The experiments performed with this latter target allowed the determination of the dimensions of the visible region with a much higher accuracy. Experience showed that the loss of intensity caused by the decrease of the thickness of the target was insignificant, a fact which probably can be explained by the multiple passing of the protons through the target. The curves represented in Fig. 2 by full lines were calculated for targets respectively 0.5 and 2.5 cm thick. The dotted curves correspond to a point target. One can see that the experimental data agree well with the calculated ones, this being a proof of the correct installation of the target.

The intensity of the γ-rays, I_{max}, at a position of the target which coincided with the collimator axis exceeded 10^4 times the intensity of γ-rays coming from the target at a position displaced 0.5 cm from the edge of the region of visibility. Therefore, the data on the intensity of γ-rays at a position of the target out of the limits of the region of visibility, are represented separately in Fig. 3 as functions of I_{max}. The experimental points represented in this figure for the normal and the reversed directions of the field H are normalized to the same γ-ray intensity, namely,the one at a position of the target on the axis of the collimator (0 on Figs. 2 and 3). From the Fig. 3 it is seen that for the normal as well as for the reversed direction of the field H, on approaching the target to the edges of the region of visibility, the intensity of the γ-rays increases at a rate independent of the particular side of the collimator axis on which the target is located. This supports the argument that the observed increase of the intensity of the γ-rays is not conditioned by the number of π^0-mesons from the decay of Λ^0-particles, but is connected with processes of emission of γ-rays by the walls of the collimator due to the electrons formed in the walls.

Let $\Delta(+x)$ be the difference in the intensity of γ-rays for the normal and reversed directions of the field H at a position of the target to the right of the collimator axis; analogously, $\Delta(-x)$ corresponds to the position of the target at left of the axis. Then, with the target out of the limits of the region of visibility, the intensity of the γ-rays, $I_{\Lambda^0}(|x|)$, caused by the decay of Λ^0-particles, will be expressed as

$$I_{\Lambda^0}(|x|) = \frac{\Delta(x) - \Delta(-x)}{2}.$$

Here we eliminate the error which could arise due to the difference in the backgrounds of the normal and the reversed field direction of H.

In Fig. 4 the ratio $I_{\Lambda^0}(|x|)/I_{max}$ for different distances of the target to the collimator axis is represented.

From the obtained experimental data it follows that, for a distance of the target in the 1.4-2.0 cm range, the average intensity of the γ-rays, $\overline{I}_{\Lambda^0}$, caused by the decay of Λ^0-particles, constitutes 0.1 ±0.4 pulses/min. The comparison of this magnitude with the intensity of the γ-rays at the position of the target on the collimator axis (I_{max} = 30,000 pulses/min) indicates that the intensity

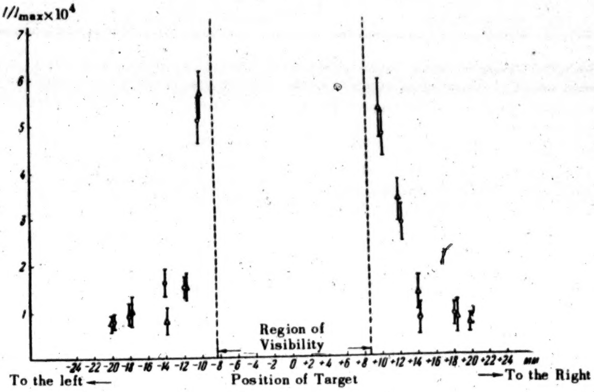

FIG. 3. The intensity of γ-rays, for a position of the target out of the limits of the region of visibility: O – for the normal, Δ – for the reversed direction of the field H.

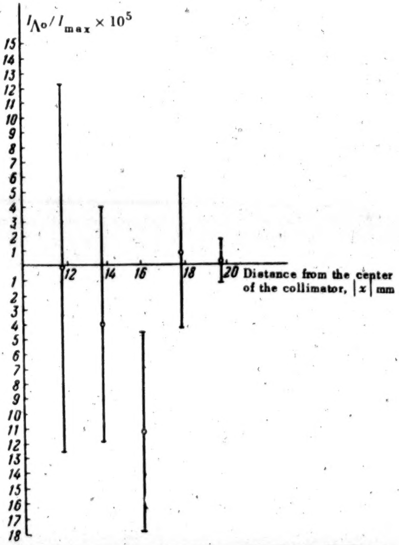

FIG. 4. Intensity of γ-rays caused by π^0-mesons from the decay of Λ^0-particles.

of rays from the decay of Λ^0-particles cannot much exceed $I_{max} \times 10^{-5}$.

The intensity I_{max} is conditioned by the decay of π^0-mesons which originate in the target. Therefore, the measurement of the creation cross section for the π^0-mesons by protons on carbon provides a possibility of determining the upper limit of the cross section $\sigma_{\Lambda^0}(\mathrm{C})$ for the creation of Λ^0-particles. The cross section for the creation of π^0-mesons by protons on carbon was found by us in a separate experiment[9], which was performed in an extracted beam of protons, the approximate value of the cross section being $30 \times 10^{-27}\,\mathrm{cm}^2$. Neglecting a small correction which takes into account the difference in angular distributions of the γ-rays originated by π^0-mesons created in the target from the π^0-mesons created as a result of the Λ^0-particle decay, we shall have:

$$I_{\Lambda^0}(|x|) = I_{max}\,\alpha(|x|)\frac{\sigma_{\Lambda^0}(\mathrm{C})}{\sigma_{\pi^0}(\mathrm{C})}. \qquad (7)$$

Here $\alpha(|x|)$ is the fraction of Λ^0-particles which decay in the field of visibility of the collimator. Assuming the mean lifetime of Λ-particles to be 3×10^{-10} sec, which in our case corresponds to a decay mean free path of ~ 5 cm, we shall have $\alpha \approx 0.2$. Hence, by Eq. (7):

$$\sigma_{\Lambda^0}(\mathrm{C}) \lesssim 10^{-30}\ \mathrm{cm}^2.$$

Thus, based on the results of our experiment, one can conclude that the cross section for the creation of Λ^0-particles by protons with an energy of 670 mev cannot be greater than $10^{-31}\ \mathrm{cm}^2$/nucleon.

DISCUSSION

The small value of the formation cross section of the Λ^0-particles in the interaction of protons with an energy of 670 mev with complex nuclei is in agreement with the hypothesis of the fundamental transformation scheme (1) of a nucleon.

Let us compare the results obtained by us with the results of Schein et al[8], where five cases of formation of Λ^0-particles by π^--mesons were observed.

The energy of the π^--meson beam in Schein's experiment was 227 mev. This value was very near to the threshold of formation of Λ^0-particles by the $\pi^- + (p, n) \rightarrow \Lambda^0 + \Lambda^0$ reaction. Having this in view, the positive results of Schein's experiment can be considered as an indication of the fact that the formation of Λ^0-particles corresponds to the virtual process $(N) \rightleftarrows (\Lambda^0) + (\pi)$. If this is true, in our work Λ^0-particles formed by the $N + N \rightarrow \Lambda^0 + N$ reaction should be observed. It must be remembered, however, that no value for the cross section for the formation of Λ^0-particles is given in the work by Schein et al, and therefore, it is too soon to affirm positively that there are contradictions between that work and ours.

It is of interest to pose a question as to what conclusion can be reached about the $N + N \rightarrow \Lambda + N$ reaction, according to the results of our experiment. It is known[4] that the cross section of formation of Λ-particles at very high energies (several thousand mev) has an order of magnitude of $10^{-27}\,\mathrm{cm}^2$/nucleon. In the paper of Chernikov[10], the average value of the cross section for the creation of pairs of Λ-particles in collisions of nucleons with an energy of 600 mev with nuclei was estimated on the assumption that the cross section of $10^{-27}\ \mathrm{cm}^2$/nucleon, found at high energies, corresponds mainly to the process of formation of two Λ-particles. In the same calculations, the momentum distribution of the nucleons in a nucleus was taken into account, and it was assumed that the matrix element does not depend on the energy. The cross section, estimated in this manner, appeared to be $10^{-29}\ \mathrm{cm}^2$/nucleon, i.e., at least 100 times greater than the value for the experimental cross section obtained by us. Having in view the arbitrary nature of the assumptions made, we can conclude that there is only a slight indication of the small probability of the $N + N \rightarrow \Lambda + \Lambda$ process at high energies as compared to the probability of the processes for the creation of Λ-particles in general. A more definite conclusion could be reached if the masses of the Λ^0-particles and the positive hyperons were equal. Data from cosmic ray work also point to the conclusion that the appearance of two Λ^0-particles is a relatively improbable phenomenon. In fact, we know of no case of simultaneous observation of two events of the V^0 type, where both could be identified as Λ^0-particle decays. On the contrary, cases of simultaneous birth of Λ^0- and θ^0-particles were observed.

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[9] B. D. Balashov, V. A. Zhukov, B. M. Pontecorvo and G. I. Selivanov, Report of the Inst. for Nuclear Problems, Acad. Sci., USSR (1954)

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The question arises as to why the $N + N \rightarrow \Lambda + \Lambda$ reaction is improbable if the scheme (1) is true? Let us examine one of the possibilities of forbiddenness/of this reaction. The large value of the probability of formation of Λ-particles in high energy collisions conduces to the idea that in creation processes of this type the hypothesis of charge independence is applicable. However, in decay processes which have a low probability the isotopic spin cannot be conserved. Therefore, on the basis of the known decay scheme, $\Lambda \rightarrow N + \pi$, one cannot conclude a priori that the isotopic spin of the Λ-particles remains a half-integer. In connection with this, one can assume that the spin of a Λ-particle is an integer. In this case it is easily seen that the reaction $N + N \rightarrow \Lambda + \Lambda$ is forbidden. If so, the isotopic spin T_θ of the heavy meson responsible for the forces between the nucleons and the Λ-particle is very probably equal to 1/2, i.e., only the positive and the neutral modality of such mesons (θ^+, θ^0) exist.

A simple method is available for a verification of the hypothesis that $T_\theta = 1/2$. It consists in comparing the number of Λ^0-particles which are formed in hydrogen and deuterium in the bombardment of π^--mesons with an energy of the order of 1000 mev. If the above-mentioned hypothesis is correct, the cross section for the formation of Λ^0-particles in *H* and *D* must have comparable values. This follows from the fact that the neutron in a deuteron cannot contribute to the formation of Λ^0-particles, because with this assumption the reaction $\pi^- + n \rightarrow \Lambda^0 + \theta^-$ would be impossible.

CONCLUSIONS

1. The cross section for the formation of Λ^0-particles in collisions of protons with an energy of 670 mev with carbon nuclei cannot exceed 10^{-31} cm^2/nucleon.

2. In nucleon-nucleon collisions, no Λ^0-particles are created according to the $N + N \rightarrow \Lambda^0 + N$ reaction.

3. The small value of the cross section for the formation of Λ^0-particles in the interaction of protons with an energy of 670 mev with complex nuclei agrees with the hypothesis of the fundamental transformation of a nucleon according to the scheme

$$(N) \rightleftarrows (\Lambda^0) + (\text{heavy meson}).$$

4. There is some indication of the fact that at high energies the reaction $N + N \rightarrow \Lambda + \Lambda$ is improbable as compared to the processes of creation of hyperons in general. This conclusion could be reached with greater conviction if the masses of Λ^0-particles and positive hyperons were equal.

The authors deem it a pleasant duty to express their gratitude to V. V. Krivitsky and A. I. Mukhin for the help rendered in the installation of the collimator, and also to the group of co-workers of the maintenance division who permitted us to perform the present work by assuring a steady operation of the synchrocyclotron.

Translated by B. Cimbleris
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SOVIET PHYSICS JETP VOLUME 2, NUMBER 1 JANUARY, 1956

The Average Number of Neutrons per Fission Event in the Photodisintegration of Uranium and Thorium

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The average number of neutrons ν from a single fission event has been measured in the photodisintegration of uranium and thorium with a mean nuclear excitation energy of 12 mev. For uranium, the value of ν is 6.2 ± 0.5, for thorium 14.2 ± 1.2. The measured quantities allowed evaluation of the relative fission probability in photodisintegration of uranium and thorium nuclei.

In the laboratory of V. I. Veksler during 1951 - 1952, the average number of photoneutrons ν from a single fission event was measured in the photodisintegration of uranium and thorium by bremsstrahlung with maximum energy $E_{max} = 18.6$ mev (average energy of nuclear excitation ~ 12 mev). The work was done with a synchroton which gave 150 γ - pulses per second. The width of a single pulse was ~ 20 μ sec. To determine the values of ν, the absolute emission of photoneutrons and fission fragments was measured, using the same number of nuclei and the same radiation dosage.

The intensity of γ radiation was measured with a thin walled ionization chamber placed between the target and the sample (Fig. 1). The chamber was connected to an integrating circuit which acted as a dosimeter.

MEASUREMENT OF THE ABSOLUTE NUMBER OF NEUTRONS

Neutrons emitted from the sample during γ irradiation were slowed in the surrounding paraffin, and registered in the ionization chamber (KN - 14) filled with BF_3. Figure 1 shows the positions of

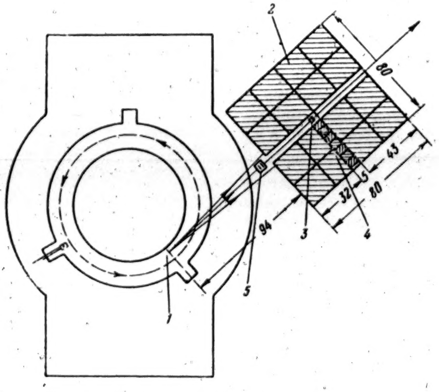

FIG. 1. Physical arrangement (dimension in cm) : *1* — internal synchroton target; *2* — paraffin block 70 cm high; *3* — sample; *4* — BF_3 ionization chamber; *5* — thin walled chamber with plane parallel electrodes for measurement of γ radiation dose (electrodes made of 50μ aluminum foil; windows in the shielding cover made of of 50 μ aluminum foil).

the paraffin block, sample, and ionization chamber. The paraffin block is 80 × 80cm, with a height of 70 cm. Samples in the form of 4cm diameter disks were placed in the center of the block. Incident γ– rays paseed through a round, 5 cm diameter, opening in the paraffin. The KN-14 chamber could be moved along a channel perpendicular to the γ - ray axis in order to measure the spatial distribution of neutron density.

Strong ionization occurred in the chamber during the passage of the γ - rays. Therefore,neutrons were registerd in the time interval between γ pulses, which followed each other every 6667 μsec. The final tube of the ionization chamber amplifier was gated after each γ pulse, passing neutron pulses into counting circuits, and closed again before the next γ pulse. By varying the beginning of the neutron counting time referred to the γ pulse, the dependence of the neutron count on time was measured. The neutron lifetime determined in this way was $\tau_0 = 183 \pm 3\ \mu$ sec.

Counting of neutrons began after a delay of 20 μ sec, and ended after 2500 μ sec. Practically all neutrons were absorbed by the paraffin during this time interval. The correction for neutrons missed in the first 20 μsec after the γ pulse was 12%. Uncertainty in the delay time did not exceed 5 μsec, and the error in neutron count in a single series of measurements did not exceed ~ 3%.

Calibration of the absolute emission of neutrons from irradiation of uranium and thorium samples was done with a Ra + Be source. The intensity of this source, measured by several methods, was $(4.69 \pm 0.27) \times 10^5$ neutrons per second.

Figures 2 and 3 show the spatial distribution of neutrons $\rho(r)r^2$ for the Ra + Be source,uranium and thorium, superimposed at the point $r = 10$ cm, where the ionization chamber was located during the measurements. The points for $r > 35$ are obtained by exponential extrapolation of the curve ends. The area under the curve $\rho(r)r^2$ for the source is $k = 1.27 \pm 0.05$ times larger than the area under the uranium curve, and $k = 1.34 \pm 0.05$ times larger than the area under the thorium curve.

Due to the large neutron background accompanying the γ - ray bundle, it was necessary to work with large samples. Sample weight for uranium was 284 gm, and for thorium, 120 gm. Neutron background was respectively ~ 10% and 20%.

In order to maintain the same neutron measuring conditions in the presence of the samples, which were strongly neutron absorbing, and in the absence of the samples during the background measurements, samples were placed in a cadmium box (wall thickness 0.5 mm). An identical empty box was placed at the sample position during background measurements. During calibration, the Ra + Be source was also placed inside such a box.

During passage through such thick samples as are necessary in this experiment, the γ - ray intensity decreases considerably, and the original energy distribution changes. The true neutron emission was obtained by extrapolating specific neutron emissions for various sample thicknesses to zero sample thickness. Both samples were made of four flat aluminum containers with an inner diameter of 4 cm, and top and bottom thickness 0.5 mm. For the uranium sample each container was filled with 71 gm of the powdered metal; for thorium four such containers were filled with pressed shavings (30 gm in each container). Neutron emission was measured when one, two, three and four containers were placed inside the cadmium box. For better extrapolation, neutron emission was also measured from sample thicknesses of one half that of a container. Fig. 4 shows the specific neutron emission for various sample thicknesses of the two materials. The specific emission obtained from the entire sample (4 containers) has been taken as unity. Specific neutron emission, extrapolated to zero sample thickness, is $1/\varkappa$, where $\varkappa = 0.59 \pm 0.02$ for uranium, and $\varkappa = 0.80 \pm 0.02$ for thorium.

Measurements of the number of neutrons from Ra + Be source in the presence of the samples showed that resonance absorption in the sample of neutrons passing through the cadmium, and fast neutron induced fission both effect the measured neutron emission by not more than one percent.

MEASUREMENT OF THE ABSOLUTE NUMBER OF PHOTOFISSION EVENTS

The absolute number of fission events was measured by observing fission fragments in an ionization chamber placed in the paraffin block instead of the sample.

To record fission fragments during the passage of γ quanta, when large ionization was produced, a differential chamber was used. The chamber had three electrodes: a central collecting, and two outside electrodes to which voltages of opposite polarities were applied. A thin layer of oxide (U_3O_8) or (ThO_2) with a diameter of 4 cm was deposited on the negative chamber electrode. Ionization impulses arising simultaneously in the identical halves of the chamber from γ quanta arrive at the amplifier grid with opposite sign, and cancel. It is thus possible to observe pulses from fission fragments. Pulses due to incomplete cancellation were several times smaller than those

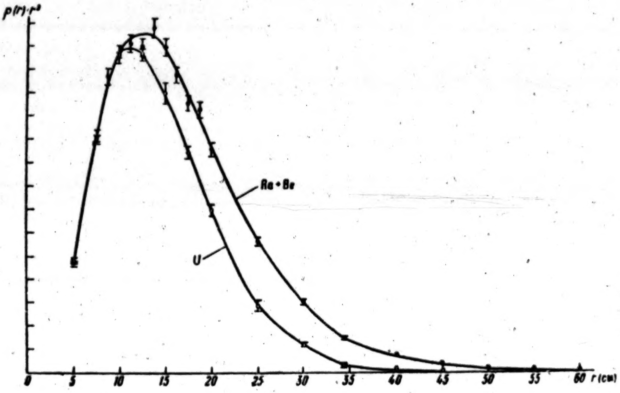

FIG. 2. Spatial distribution of neutrons from uranium.

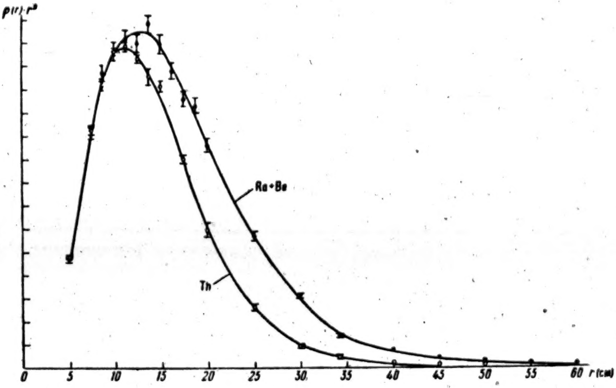

FIG. 3. Spatial distribution of neutrons from thorium.

due to α-particles emitted by the uranium and thorium layers. These pulses were cut off during the fission fragment count.

The chamber wae filled with chemically pure argon at a pressure of 1.5 atmospheres, and utilized the electron part of the pulse, which insured low noise and an absence of microphonic effects. The electrodes had 7 cm diameter in order that fragments emitted from the edge of the layer would not leave the confines of the chamber and be missed.

An electronic circuit counted pulses which were > 2.5 times the maximum α-particle pulse.

The oxide layers used in the measurements were prepared by baking water solutions of uranium nitrate and thorium in nitric acid. The deposition was on aluminum foil in the course of several hours at 500°.

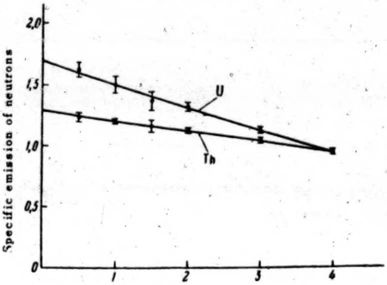

FIG. 4. Relative emission of photoneutrons during irradiation of various thickness targets. The abscissa is marked in thicknesses of 1, 2, 3, 4 containers. The thickness of each container with uranium was 5.65 gm/cm^2, and with thorium, 2.38 gm/cm^2.

The uranium measurements were carried out with a single layer whose weight was determined by the number of α-particles emitted. For the layer used, $w = 1.78 \pm 0.02$ mg, which corresponded to a thickness of 0.17 mg/cm^2 of the oxide. Thorium measurements were made with two layers: $w_1 = 1.38 \pm 0.04$ mg; $w_2 = 2.36 \pm 0.07$ mg. The weight of the thorium layers was determined from the amount of solution of known concentration which was deposited on the surface used in the measurements. The solution concentration was found by weighing the ThO_2 produced after slow evaporation and subsequent baking of a known volume of the solution. A correction to the fission fragment count was made for each layer. Since the areal density of a layer was of the order of several tenths mg/cm^2, the correction to the fission fragment count was not over several percent.

During γ irradiation of the layers, various fission events could have been caused by background neutrons or photoneutrons emitted from the ionization chamber walls. Measurement of fission fragments during the bombardment of the layers by neutrons from the Ra + Be source showed that the neutron induced fission events did not exceed 0.2% of the total number of registered fissions.

RESULTS

The average number of neutrons ν emitted per photofission event was obtained: for uranium 6.2 ±0.5; for thorium 14.2 ±1.2. The value of ν for thorium is the average of two values $\nu_1 = 14.1 \pm 1.1$ and $\nu_2 = 14.4 \pm 1.3$ which were obtained for the two thorium layers.

The measured values of ν allow evaluation of the photofission probability in uranium and thorium. During irradiation of these nuclei with γ-rays of $E_{max} = 18.6$ mev, four reactions involving photoneutrons are energetically possible:

$$(\gamma, f), \ (\gamma, n), \ (\gamma, nf) \quad \text{and} \quad (\gamma, 2n).$$

The following Table shows the various reaction thresholds[1]. The probability for photoproton emission for nuclei of $Z > 80$ is a few tenths of a percent. The probability for a (γ, γ') reaction should not exceed several percent[2].

Table

Reaction*	U*** (MeV)	Th*** (MeV)
(γ, f)	5.08±0.15	5,40±0,22
(γ, n)	5.97±0.10	6.35±0,04
(γ, 2n)	11,4±0.2	11.85±0,2
(γ, nf)	11.6±0.3	12.15±0.3

*Experimentally determined thresholds for (γ, n) and (γ, f) are given[1].

Thresholds of (γ, nf) and (γ, 2n) are the sum of the experimentally determined first neutron binding energy, and the calculated second neutron binding energy and fission threshhold of U^{237} and Th^{231}.

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[2] A. G. W. Cameron and L. Katz, Phys. Rev. **84**, 608 (1951)

Thus at the present excitation energies, photofission and photoneutron emission are essentially the only competing processes of uranium and thorium photodisintegration.

The fraction of nuclei α which fission as a result of (γ, f) and (γ, nf) reactions is $(\bar{\sigma}_{\gamma,f} + \bar{\sigma}_{\gamma,nf})/\bar{\sigma}_{\gamma}$; the fraction of nuclei β which decay by the emission of one or two neutrons is $(\bar{\sigma}_{\gamma,n} + \bar{\sigma}_{\gamma,2n})/\bar{\sigma}_{\gamma}$ where $\bar{\sigma}_{\gamma}$ is the average absorption cross section for the effective quanta; this is equal to the sum of the average cross sections of all competing reactions $(\bar{\sigma}_{\gamma,n} + \bar{\sigma}_{\gamma,f} + \bar{\sigma}_{\gamma,nf} + \bar{\sigma}_{\gamma,2n})$.

The measured quantity $\nu = \nu_0 + \bar{n}\,(\beta/\alpha)$, where ν_0 is the average number of neutrons emitted from the nucleus undergoing fission and $\bar{n}$ the average number of neutrons emitted in the photodisintegration of a nucleus through the reactions (γ, n) and (γ, $2n$).

In agreement with the γ quantum energy dependence of the photofission cross section obtained by I. V. Chuvilo, the average excitation energy of fissioning uranium and thorium nuclei irradiated by bremsstrahlung of $E_{\max} = 18.6$ mev is ~ 12 mev. With such excitation the average number of neutrons accompanying fission is around three. From the known reaction yields of (γ, $2n$) for bismuth and tantalum[3], and also from calculations based on statistical theory, $\bar{n} \approx 1.2 - 1.4$. Substituting these values for ν_0 and $\bar{n}$ into the expression for ν, we obtain:

$$\left(\frac{\beta}{\alpha}\right)_U = \left(\frac{1-\alpha}{\alpha}\right)_U = \frac{3.2}{\bar{n}} \approx 2.7 \div 2.3,$$

$$\alpha_U \approx 0.27 \div 0.30;$$

$$\left(\frac{\beta}{\alpha}\right)_{Th} = \left(\frac{1-\alpha}{\alpha}\right)_{Th} = \frac{11.2}{\bar{n}} \approx 9.3 \div 8.0.$$

$$\alpha_{Th} \approx 0.10 \div 0.11.$$

The values obtained for α_U and α_{Th} give an upper limit to the average fission probability of an excited U^{238} or Th^{232} nucleus, since α includes fission events resulting from the (γ, nf) reaction, where the nucleus remaining after emission of the first neutron undergoes fission.

The above values show that during the absorption of 12 mev γ quanta, the fission probability of uranium is ~ 1/4 to 1/5 and the fission probability of thorium is ~ 1/10. These agree with similar evaluations made by other authors[4].

The average absorption cross sections for γ quanta by uranium and thorium nuclei do not differ by more than a few percent[5], and therefore the ratio α_U / α_{Th} must be equal to the ratio of the photofission cross sections of these elements. The value obtained for the ratio $\alpha_U / \alpha_{Th} = 2.7$ is in good agreement with results given in other work[6].

There is only one measurement of ν in the literature, where a similar method was used to measure ν for uranium for $E_{\max} = 23$ mev[7]. The value obtained, $\nu = 10.5 \pm 2$, contradicts all available data for the average photofission probability of uranium. No reports of measurements of ν for thorium have appeared in the literature.

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Translated by G. L. Gerstein
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Asymmetry of Fragment Ranges in the Fission of Heavy Nuclei by Ultrafast Particles

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From the data of a study of fission of U, Bi and W by 660 mev protons, carried out in fine-grained nuclear P-9 emulsions, we have constructed the distribution of the ratio of fragment ranges for various excitation energies of the fissioning nucleus. We find that with increasing excitation energy the fraction of strongly asymmetric fissions increases; this increase is significantly greater for Bi than for U. For excitation energies $\gtrsim$ 400 mev, the character of fission is the same for U, Bi and W. The region of excitation energy in which fission is most symmetric is estimated to be 60-100 mev for uranium, and ~ 100 mev for Bi.

INTRODUCTION

THE distribution of fragment masses from the fission of heavy nuclei by slow neutrons is a double-peaked curve, indicating the asymmetric character of the fission. With increasing energy of the particles producing the fission, the mass distribution becomes more symmetric, so that for 90 mev neutrons the spectrum of fragment masses has one sharp maximum.

Thus, in this region of excitation energy of the U nucleus, the role of the symmetric form of fission increases with increasing excitation. The change in the character of fission for still greater excitation energies is extremely interesting and is of importance for clarifying the mechanism of nuclear fission at high excitation energies.

Unfortunately, radiochemical methods can be applied to this problem only in the range of energy of the incident particle for which the fissioning nucleus is completely black, since only then can we identify the excitation energy of the nucleus with the energy of the particle. In the high energy range a given energy of the incident particle will give a whole spectrum of nuclei at various excitation energies, and only an insignificant number of nuclei will have excitation energies equal to the kinetic energy of the particle. In applying radiochemical methods to study the fission products, we can only get some average excitation energy, which depends slightly on the particle energy in the region where the nucleus is transparent, so that it is difficult to detect any change in the character of fission which may exist.

An attempt has been made in the present work to determine the character of fission of heavy nuclei as a function of excitation energy for high excitation (>100 mev).

EXPERIMENTAL DATA

As a first attempt to solve the problem, we have used the method of thick-layered emulsions. Type P-9 nuclear plates, loaded with uranium, bismuth and wolfram by soaking in appropriate aqueous solutions, were irradiated with a 660 mev proton beam. The developed plates were searched under a microscope to study cases of fission of these nuclei. The fission of heavy nuclei by protons of a given energy is frequently accompanied by the emission of several charged particles; as shown in reference 1, the emergence of a certain number of charged particles corresponds to a definite average angle between the fragment directions, and consequently to a definite excitation energy of the nucleus. Therefore, if we classify cases of fission according to the number *n* of charged particles emitted during fission, we automatically select fissioning nuclei with a given excitation energy.

Figure 1 shows the distribution in range in emulsion of individual fragments from the fission of U, as a function of the excitation energy of the fissioning nucleus (~ 200 fragments were measured in each group). Distributions are shown for $E_{\text{exc}} \approx 60$, 240 and 540 mev together with the range distribution for thermal fission of U, for comparison.

From consideration of Fig. 1, we can arrive at the following conclusions:

1. The most probable range of the fragments decreases with increasing excitation energy; this is apparently related to the decrease in kinetic energy of the fragments because of the lowering of the *Z* of the nucleus before fission.

2. Unlike the fission by slow neutrons, the range distribution of the fragments shows a single well-defined maximum, but the half-width of the distribution curve increases substantially as we go

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[1] V. P. Shamov, Otch. RIAN (Report of Radium Inst., Acad. Sci., USSR) 1954

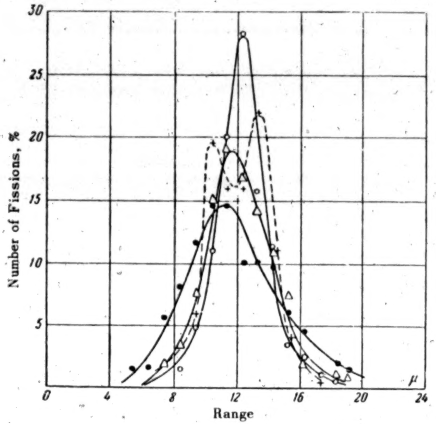

FIG. 1. Range distribution of individual fragments from uranium fission: + – fission of U^{235} by thermal neutrons, ○ – fission of uranium for $E_{exc} \approx 60$ mev, Δ – fission of uranium for $E_{exc} \approx 240$ mev, ● – fission of uranium for $E_{exc} \approx 540$ mev.

to high excitation energies.

Since this increase in half-width of the fragment range distribution suggests an increase in the fraction of asymmetric fissions with increasing excitation energy, we have studied the distribution of the ratio of the ranges of the two fragments for various excitation energies in U, Bi and W. The Table and Figs. 2 and 3 show the results of measurements of the ratio l_l/l_h (l_h, l_l are the ranges of the heavy and light fragments, respectively) for various excitation energies of U and Bi, as well as for the fission of W.

The Table also gives the distribution of the ratio of fragment ranges for fission of U^{235} by slow

Ratio of Ranges	Fraction of fissions with given l_l/l_h, in %							
	U			Bi			W	U^{235}
	$n = 0$ $E_{exc} \approx 60$ mev	$n = 2$, $E_{exc} \approx 240$ mev	$n > 3$, $E_{exc} \approx 380$ mev	$n = 0; 1$, $E_{exc} \approx 150$ mev	$n = 2$, $E_{exc} \approx 240$ mev	$n > 3$, $E_{exc} \approx 380$ mev	$n > 1$, $E_{exc} \approx 400$ mev	Thermal Neutrons
1.0—1.15	46	32	28.5	54.5	38.3	29.8	32	28
1.15—1.30	32	27.2	25.6	23.6	28.0	17.3	21	45.5
1.30—1.45	8	19.4	18.6	8.2	14.7	22.0	15	17.3
1.45—1.60	10	10.7	8.4	7.25	7.35	11.5	13	4
1.60—1.75	2	5.8	6.4	2.70	7.35	9.6	4	4
1.75—1.90	3	1.0	2.8	3.60	2.94	1.92	4	1.34
1.90—2.05	—	1.0	4.6	—	1.47	0.96	2	—
2.05—2.20	—	2.0	1.46	—	—	2.88	2	—
2.20—2.35	—	—	0.3	—	—	1.92	3	—
2.35—2.50	—	1.0	1.27	—	—	1.96	2	—
2.50—2.65	—	—	0.99	—	—	—	1	—
2.65—2.80	—	—	0.67	—	—	0.96	—	—
2.80—2.95	—	—	0.16	—	—	—	—	—

neutrons, observed in the same type of emulsion.

DISCUSSION OF EXPERIMENTAL DATA

From the Table and Figs. which have been presented, we can draw some very important conclusions if we are convinced that the features of the distributions of ranges and range ratios observed in the experiment are related to the mass distribution of the fragments. A basis for such a conclusion is that fact that for fission of U by thermal neutrons the range distribution of the fragments mimics the mass distribution of the fragments, so that asymmetry of fission with respect to fragment ranges can signify only an asymmetry of fission with respect to fragment masses. Moreover, in all cases the spectrum of fragment ranges has a less marked dependence, so that we may hope that the changes observed in the spectra of fragment ranges occur even more markedly in the spectra of fragment masses*.

The dependence of l_1/l_h shown in Figs. 2 and 3 is very interesting. From analysis of these data we can draw the conclusion that with increasing excitation energy (i.e., energy transferred to the nucleus by the 660 mev proton) the fraction of strongly asymmetric fissions increases for both U and Bi. This applies to excitations > 60 mev for uranium, and > 150 mev for bismuth. If we draw the same distributions for fixed excitation energy and for the various nuclei (Figs. 4, 5, 6) we can reach the following conclusions:

1. For not too high excitation energy (~ 100 mev), the fraction of symmetric fissions is greater for bismuth than for uranium.
2. As the excitation energy is increased, the character of fission undergoes a more marked change in bismuth than in uranium.

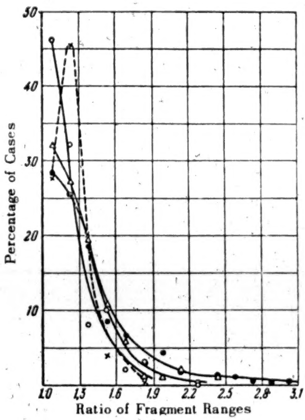

FIG. 2. Distribution of the ratio l_1/l_h of ranges of fragments from fission of U for various excitation energies: O – $E_{exc} \approx 60$ mev, Δ – $E_{exc} \approx 240$ mev, ● – $E_{exc} \approx 380$ mev, × – fission by thermal neutrons.

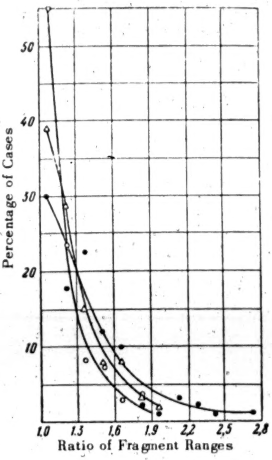

FIG. 3. Distribution of ratio l_1/l_h of ranges of fragments from fission of Bi for various excitation energies: O – $E_{exc} \approx 150$ mev, Δ – $E_{exc} \approx 240$ mev, ● – $E_{exc} \approx 380$ mev.

3. For excitation energies ~ 400 mev, the character of fission is the same for all the nuclei, U, Bi and W.

In order to convince ourselves that the observed asymmetry in fragment ranges was not caused by the

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* Similar conclusions can be drawn from reference 2, where it was shown that in 96% of the cases of uranium fission, longer range was associated with smaller mass.

[2] R. Mathieu and P. Demers, Canad. J. Phys. 31, 97 (1953)

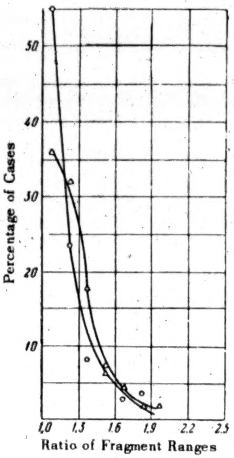

FIG. 4. Distribution of ratio of fragment ranges for fission of U and Bi, for initial excitation energy ≈ 150 mev: O – Bi. Δ – U.

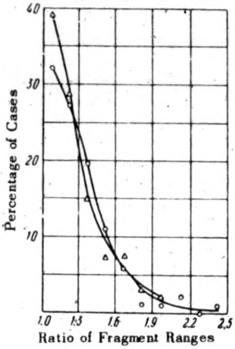

FIG. 5. Distribution of ratio of fragment ranges for fission of U and Bi, for initial excitation energy ≈ 240 mev: O – U, Δ – Bi.

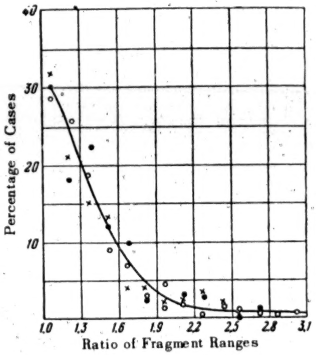

FIG. 6. Distribution of the ratio of fragment ranges from the fission of U, Bi and W for excitation energy ≈ 400 mev: O – U, ● – Bi, × – W.

translational velocity of the fissioning nucleus, we calculated the resulting asymmetry for maximum velocity of the uranium nucleus (excitation energy ≈ 600 mev), assuming that the fission is symmetric. It turned out that the maximum possible asymmetry in this case was $l_I/l_{II} = 1.13$, which could by no means account for the distribution of l_l/l_h. In addition, if there were actually an effect of the translational velocity on the asymmetry in fragment ranges, then it would be difficult to explain the large effect of the velocity on the range distribution of fragments from uranium fission and the small effect on the range distribution from the lighter Bi nucleus, whose velocity is larger for equal excitation energy.

These remarks lead to the conclusion that at high excitation energy an asymmetric form of nuclear fission begins to play an important role. This increase in the role of strongly asymmetric fissions with excitation energy can be expressed very roughly in terms of the variation of the ratio of fissions with $l_l/l_h \geq 1.45$ to fissions with $l_l/l_h \leq 1.15$ (cf Fig. 7). From the graph we can make a crude estimate of the range of excitation energy in which the fission of U and Bi is most symmetric. We may expect that this region will occur in the range 60-100 mev for uranium, and close to the threshold for emissive fission, i.e., ~ 100 mev for bismuth.

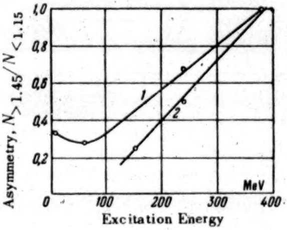

FIG. 7. Dependence of asymmetry of fission of U and Bi on excitation energy: *1*. U, *2*. Bi.

In conclusion we should point out that none of the existing hypotheses for explaining the character of nuclear fission fits the present case. In addition, one is obliged to assume that the character of fission is determined, not by the excitation energy remaining in the nucleus at the moment of fission after the emission of particles, but rather by the initial energy of excitation of the nucleus as a result of its interaction with the proton. This makes it necessary to study these questions further.

The authors take this opportunity to express their thanks to Prof. N. A. Perfilov for participating in discussion of the results of the present work.

Translated by M. Hamermesh
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The Angular Distribution of Fragments from Fission of Uranium at High Excitation Energies

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J. Exper. Theoret. Phys. USSR **29**, 292-295 (September, 1955)

We have investigated the fission of uranium in P-9 fine-grained nuclear emulsions, soaked in aqueous solutions of uranium salts and irradiated with 660 mev protons. The angular distribution of fission fragments relative to the direction of the proton beam was studied at excitation energies of the uranium nucleus ~ 75, 150 and 300 mev. The angular distribution of the fragments can be described approximately by the function $a + b \sin^4 \varphi$. The anisotropy increases somewhat with increasing excitation energy.

INTRODUCTION

THE presently available experimental data on angular distribution of fragments from the fission of heavy nuclei show that there is an appreciable anisotropy in the distribution of fragments relative to the direction of the beam of particles producing the fission. The character of this anisotropy depends essentially on the type of particle interacting with the nuclei.

Already in 1953 it was shown, in work carried out in our laboratory[1], that there is an appreciable anisotropy in the angular distribution of fragments from fission of uranium by 460 mev protons. The ratio of the number of fragments in the direction of the beam and perpendicular to it is substantially less than unity. Roughly the same type of anisotropy was observed in the photofission of thorium[2]. In this case the angular distribution can be expressed as

$$I(\varphi) = a + b \sin^2 \varphi,$$

where φ is the angle measured from the direction of the gamma ray beam.

Later there appeared reports in the literature of investigations of the angular distribution of fragments from fission of uranium and thorium by neutrons from thermal energy up to 20 mev, and by 22 mev protons. It was shown[3] that for fission of thorium by 22 mev protons the angular distribution is satisfactorily described by a formula of the type $I(\varphi) = a + b \cos^2 \varphi$, where the ratio b/a increases with increasing ratio of mass of the fission fragments. In addition, the dependence of the angular distribution of fission fragments on the energy of the incident neutrons was investigated[4]. For fission by neutrons with energy 10^{-7}; 2.5; 4.6; 7.5; 14.3; and 20.4 mev, the ratios of yields of fragments at angles 0° and 90° were: 0.99; 1.02; 1.13; 1.36; 1.27; 1.11. Thus, as the neutron energy is increased in this range, the anisotropy first increases and then decreases, but for all energies in this interval the angular distribution is described by the formula

$$I(\varphi) \sim (1 + a \cos^2 \varphi + b \cos^4 \varphi).$$

In the present work we studied the angular distribution of fragments from fission of uranium by 660 mev protons.

EXPERIMENTAL DATA

The angular distribution of fragments from fission by 660 mev protons was studied using thick-layered emulsions. Fine-grained nuclear emulsions, type P-9, (prepared in the laboratory of N. A. Perfilov), were soaked in a solution of uranium salt and irradiated with a beam of protons. The developed plates were searched for cases of fission of U, and the projected angles between the direction of emergence of the fragments and the direction of the proton beam were measured. Then, to separate cases of fission with a definite excitation energy, the number of charged particles accompanying the fission was counted. As was shown previously[5], a given number of charged particles from fission is associated with a definite average angle between the two fragments, and consequently, with a definite excitation energy of the fissioning nucleus.

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[1] V. I. Ostroumov, Otch. RIAN (Report of Radium Inst., Acad. Sci., USSR) 1953

[2] E. J. Winhold et al, Phys. Rev. 87, 1139 (1952)

[3] B. I. Cohen et al, Phys. Rev. 94, 625 (1954)

[4] J. E. Brolley et al, Phys. Rev. 95, 651 (1954)

[5] V. P. Shamov, Otch. RIAN (Report of Radium Inst., Acad. Sci., USSR) 1954

Table I gives the results of measurement of the angular distribution of uranium fission fragments for high excitation energies.

TABLE I

Angle interval (degrees)	Number of fissions accompanied by n charged particles		
	$n=0$, $E\sim 60$ mev	$n=1$, $E\sim 150$ mev	$n\geq 2$, $E\sim 320$ mev
0—10	114	59	85
10—20	119	60	81
20—30	124	61	63
30—40	130	60	70
40—50	123	70	93
50—60	123	77	96
60—70	134	73	90
70—80	139	77	104
80—90	132	87	114

Figure 1 is a histogram of the angular distribution of all observed cases of fission.

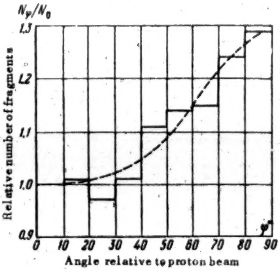

FIG. 1. Angular distribution of uranium fission fragments with respect to the direction of the 660 mev proton beam. The dotted curve is the function $N_\varphi/N_0 = 1 + 0.29 \sin^4 \varphi$

DISCUSSION OF RESULTS

If we try to describe the angular distribution shown in Fig. 1 by a function of sin φ, then we see that the function must contain at least the fourth power of sin φ. The dotted curve in Fig. 1 shows the variation $N_\varphi/N_0 = 1 + 0.29 \sin^4 \varphi$.

With increasing excitation energy, the angular distribution changes very slowly, and the anisotropy increases somewhat. If we express the anisotropy as the ratio of the number of fissions in the interval 60-90° to the number of fissions in the interval 0-30°, we get the following variation in the anisotropy (Table II).

TABLE II

E_{exc} (mev)		~60	~150	~320
Anisotropy	$\frac{N_{\varphi\geq 60°}}{N_{\varphi<30°}}$	1.13 ± 0.1	1.31 ± 0.15	1.35 ± 0.16

All our statements refer to the distribution of projections of fission fragment tracks on the plane of the emulsion. As for the spatial distribution of the fragments, we should point out that the anisotropy of the distribution per unit solid angle, along the beam and perpendicular to it, will be even somewhat greater than the anisotropy in the distribution of the projections.

If we look at the whole range of nuclear excitation energies for irradiation with nucleons, we can plot the dependence of the anisotropy of the distribution on excitation energy (Fig. 2).

We find it difficult at present to explain either the shape of the angular distribution of fragments or its dependence on excitation energy. There undoubtedly is some explanation for the fact that there is a well-defined narrow range of energy within which the anisotropy of the fission fragment distribution reverses its character. We should mention another important fact. As shown by other work[6], there is a definite connection between the excitation energy of the uranium nucleus and the range distribution of the fragments. With increasing excitation energy of the uranium nucleus, the asymmetry in the distribution of fission fragment ranges increases. Thus, simultaneously with the increase in anisotropy of fission relative to the direction of the incident proton beam there is an increase in the contribution of the asymmetric form of nuclear fission. This variation is clear from Table III.

In conclusion, we should note that one may speculate that the observed anisotropy in the angular distribution of fission fragments and its

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[6] V. P. Shamov and O. V. Lozhkin, Otch. RIAN (Report Radium Inst., Acad. Sci., USSR) (1955); J. Exper. Theoret. Phys. USSR 29, 286 (1955); Soviet Phys. 2, 111 (1956)

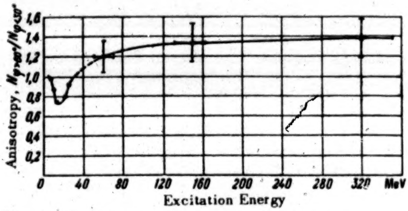

FIG. 2. Dependence of the anisotropy in distribution of fragments from fission of uranium on the excitation energy of the nucleus. The initial part of the curve is taken from the data of reference 4.

TABLE III

E_{exc} (mev)	~60	~150	~320
Anisotropy in angular distribution: $\frac{N_{\varphi>60°}}{N_{\varphi<30°}}$	1.13	1.31	1.35
Asymmetry in range of fragments: $\frac{N(l_l/l_h>1.45)}{N(l_l/l_h<1.15)}$	0.28	0.45	0.86

relation to the asymmetry of fission may be directly connected with the very mechanism of the fission process, and therefore deserves most careful study.

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The Characteristics of "Specific" Zones

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Twenty characteristics distinguishing the "specific" zones from the rest of the nuclide system are listed. Almost all characteristics are substantially different for odd and even Z. The author's opinion is that the specific zones cannot be considered as regions influenced by the "magic" numbers; on the contrary, the "magic" numbers are considered a consequence of the specific zones.

IN 1947 I pointed out[1] specific regions (zones) in which the atomic numbers Z have many properties that distinguish them from all other Z numbers. The first such property is that the odd Z of these regions each have two isotopes. This first distinction vanishes if the atomic nuclei are characterized by the A number rather than the Z number. In this case, however, there are several nuclides having the same A number (two and sometimes even three), but they do not fall in any specific zones. Apparently, it is sensible to assume that Z (the proton number) is the principal characteristic of the nucleus, for there can be no atomic nucleus without any protons. Although the neutron, in accordance with modern representations, is an equally valid component of the atomic nucleus, we nevertheless do not know of any real nuclide consisting of neutrons alone. An atom of matter exists only if it contains a proton. Following Cherdyntsev's initiative, Goeppert-Mayer and Teller advanced a hypothesis concerning polyneutron bodies, but such were never observed. The number Z establishes not only the chemical, but also the physical, nature of matter.

The regions referred to are the Z-number intervals from 16 to 20 (first specific zone); from 28 to 38 (second specific zone, which is, however, more complicated in character) from 46 to 52 (or 54) (third specific zone), from 58 to 62 (or 64) (fourth, not clearly defined) and from 72 to 82 (fifth specific zone, also complex). I tried to prove in several articles certain specific properties of these zones. However, my attempts to highlight them apparently were not recognized by other physicists. Recently, I found still another set of features that distinguish these Z regions and I am listing below all the features, both old and new, in order to exhibit more clearly the role of these regions in the formation of the nuclei.

1. The first feature, as was already indicated, is the existence of two isotopes for odd Z. However, the specific zones extend somewhat beyond this region and include also the neighboring even Z. In Fig. 1, a dotted line passing through all the curves marks the specific zones. The fourth zone, for which no complete data are available, is not marked.

2. The second distinct feature is the discontinuity in the number of free neutrons of the most abundant isotope (see reference 1). It is very clearly pronounced up to $Z = 43$, and then becomes less and less distinct.

3. The third remarkable feature is the symmetric distribution of the odd nuclei with two isotopes about the above-mentioned discontinuity in the number of free neutrons.

4. The fourth feature is the accumulation of the maximum number of odd-Z isotopes in this region. This is shown by curve *I* of Fig. 1.

5. Curve *II*, showing $(N_{max} - Z)/Z$ for the odd nuclei corroborates the fourth feature. Curve *II* has maxima at the same points as the first curve.

Curve *II'* shows $(N_{min} - Z)/Z$ versus Z. It also points to a certain connection with the specific zones, although not as clearly as curve *II*. On the other hand, curve *III*, which gives $(N - Z)/Z$ for the most abundant isotopes, does emphasize feature No. 2 and accentuates the specific zones very sharply.

A characteristic feature of the nuclide system is the clearly pronounced pairing of the nuclear particles. Nuclei with even numbers of protons play a predominant role among the nuclei. The neutrons occur mostly in even numbers. Even Z have predominantly even N, odd Z always have

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[1] M. A. Levitskaia, Dokl. Akad. Nauk SSSR 55, 399 (1947)

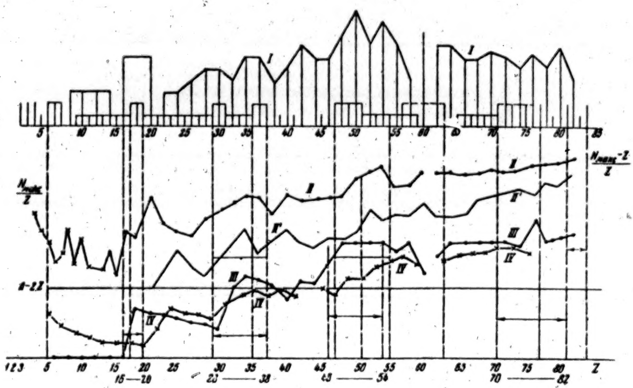

FIG. 1. Curve *I* -- number of stable isotopes versus *Z*; *II* -- $(N_{max} - Z)/Z$; *II'* -- $(N_{min} - Z)/Z$; *III* -- $(N - Z)/Z$ for maximum abundance isotopes; *IV* -- number of neutron pairs divided by number of proton pairs.

even *N*, except for the La^{138}_{57} nucleus which turns out to be naturally radioactive. We shall discuss only values of *Z* for the stable nuclides, that is, we shall consider *Z* up to 84. The very close relationship between the proton and the neutron is universally known. This connection is also clearly exhibited in a system in which the nuclei become more complicated. We shall therefore pay attention to the pairing feature. This points to a sixth characteristic:

6. The ratio between the number of neutron pairs and the number of proton pairs also reaches a maximum in the specific zones (curve *IV*). This feature is naturally not independent but is a corollary of the fifth one.

Of interest in connection with the pairing of particles is the existence of odd *N*. This is observed only in conjunction with even *Z*. If we follow the sequence of nuclei having even *Z*, we see that a nucleus having odd *N* occurs once for all *Z* up to 48, except $Z = 18$, in which there is no such nucleus, and $Z = 42$, which has two such nuclei, evidently because of the neighboring unstable $Z = 42$. Every even-*Z* nucleus above $Z = 48$ has two odd-*N* isotopes, except for $Z = 50$ and $Z = 60$, in which there are three such nuclei, $Z = 68$ which has none, and $Z = 66, 74, 73$ and 82, which have one each. No specific zones are indicated with respect to the number of nuclei with odd *N*. However, if we list the values of odd-*N* in sequence, they behave in a regular fashion up to $Z = 16$, starting with unity (at $Z = 2$); at $Z = 16$ there is a discontinuity of 6, and further on we observe a discontinuity by 4 in the specific zones.

7. The seventh feature are the discontinuities in the consecutive even *N* (even nuclei) in the specific zones.

The discontinuities result in dropping out of several odd *N*, which generally are not found in the stable nuclei at all: 19 and 21 in the first specific zone, 35, 39 and 45 in the second, 57 and 61 in the third, 89, 115 and 123 in the fifth. This points to the following feature:

8. The specifc zones contain values of odd *N* that do not correspond to a single stable nucleus.

The fact that the greatest number of isotopes and the highest values of *N*/*Z*, and with this also of *A*/*Z*, are located in the specific zones, indicates that the nuclei have a maximum lifetime (maximum stability) in these zones. American investigators have introduced the concept of magic numbers of proton and neutrons. These numbers (20, 28, 50, 82 and 126) are the upper bound of the specific zones, with the exception of 28, which is a lower bound. As far as the number of neutrons goes, 20 and 28 are in the first zone, 50 is in the second, 82 is in the fourth and 126 is at the upper end of the fifth. The American investigators point at a minimum effective capture neutron cross section σ_t as being the principal feature of nuclear stability. Evidently this applies only to the slowest, thermal neutrons. The effective cross section depends strongly on the neutron energy and at higher energies we encounter cases of resonance or near-resonance. If the specific zones are regions of more stable nuclei, the minimum of σ_t should not be sharp, but sufficiently diffuse in the specific zones. It is quite difficult to plot the effective cross section as a function of Z, since different isotopes having the same Z may have different values of σ_t. Nevertheless, if one of the isotopes have a very small σ_t the total σ_t may be considerably reduced. Table 2 on page 13 of reference 2 lists σ_t for individual isotopes of various elements obtained by thermal activation.

Reference 3 contains a table of σ_t for thermal neutrons and for rapid (1 mev) neutrons. At thermal energies there are such large discontinuities in σ_t from one value of Z to the next, that it becomes necessary to plot separate curves for even and odd Z (and most ordinates have several points each at that). Curve V of Fig. 2 shows that for even Z, up to $Z = 22$, the values of σ_t are generally small, and that the minimum of σ_t is observed in Ti^{50}_{22} (isotope of low abundance), and not in Ca^{40}_{20}. This is followed by a sharp rise; a decrease starts at $Z = 28$ and we have a minimum value at $Z = 32$ to 38. A rise occurs again at $Z = 40$. A decrease starts at $Z = 45$ and small values are observed up to $Z = 58$. The minimum values are found for $Z = 50$ and 54. In regions of larger Z (there are no values up to $Z = 70$) we have extremely large values for Z from 70 to 80, which are followed by a reduction at $Z = 78$. At $Z = 82$ and 83, we have a minimum.

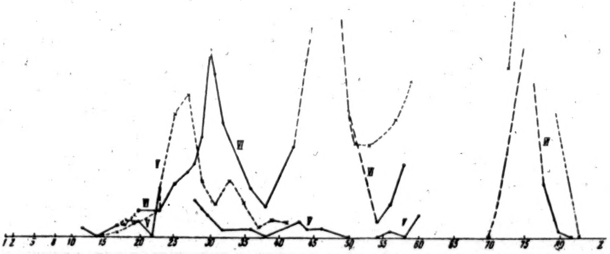

FIG. 2. Curve V -- σ_t for even (points) and odd (crosses) values of Z at thermal energies; VI -- ditto for 1 mev energy.

According to the data of the same reference the values of σ_t are generally small at 1 mev up to $Z = 20$, with a minimum at $Z = 14$ (curve VI). At $Z = 20$, a rise begins; at $Z = 28$, it is already considerable, and at $Z = 30$, we have a maximum. A decrease starts at $Z = 32$ with a minimum at $Z = 38$, followed by a subsequent decrease starting at $Z = 50$; a minimum is reached at $Z = 54$. The value is again very large in the region beyond $Z = 70$, but a decrease starts again at $Z = 80$.

I plotted σ_t from Goodman's Tables for 10 mev neutrons (curve VII, Fig. 3). He apparently gives total values of σ_t. A minimum occurs at $Z = 16$. From $Z = 16$ to $Z = 24$, there are no values for even Z. A rise starts at $Z = 24$ and a maximum occurs at $Z = 28$ (but there is no resonance here). From $Z = 30$ to $Z = 36$, we observe relatively small values of σ_t. A minimum occurs also at $Z = 50$, and an even lower one at $Z = 54$. A rise starts at $Z = 54$, and a maximum occurs at $Z = 76$. A decrease begins at $Z = 78$, but a minimum is reached at $Z = 80$; $Z = 82$ yields a large value of σ_t.

Reference 4 gives a rather large long table of

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[2] C. Goodman, *Effective Neutron Cross Sections of Elements*, 1948

[3] D. J. Hughes, R. C. Garth and J. S. Levin, Phys. Rev. 91, 1423 (1953)

[4] R. H. Hildebrand and C. E. Leith, Phys. Rev. 80, 842 (1950)

σ_t for various Z at 42 mev. The curve plotted from this table (curve *VIII*, Fig. 3) no longer exhibits the considerable discontinuities between even and odd nuclei. This curve shows clearly a smooth reduction in σ_t from $Z = 28$ to $Z = 38$ and from $Z = 46$ to $Z = 54$. The behavior of σ_t for the odd Z is approximately the same. There are undoubtedly valid grounds for stating the existence of the following features:

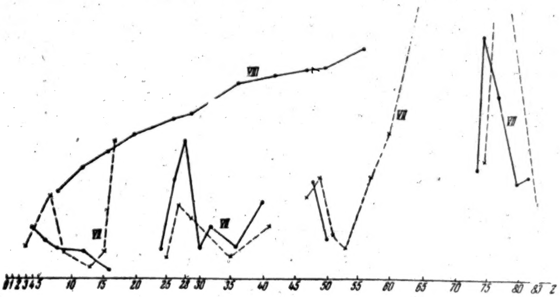

FIG. 3. Curve *VII* -- σ_t versus Z for neutrons with 10 mev; *VIII* -- ditto for 42 mev.

9. The effective capture cross sections become smaller in the region of the specific zone.

10. The magic Z numbers lie on the boundaries of the specific zones, while the magic N numbers lie at the upper bounds of the specific zones.

Let us now proceed to the radioactive properties. Four of these have already been indicated in my earlier works:

11. Naturally radioactive nuclei occur at the boundaries of the specific zones.

12. The Shchukarev-Mattauch pairs (isobars of neighboring Z) are found in the specific zones.

13. The longest half-lives of the isotopes next heavier than the isotopes of the stable nuclei are located at the boundaries of the specific zones[5]. This is directly related to properties 11 and 12.

For even nuclei the longest half-lives of the plus-active nuclei lie also in the specific zone (curve *IX*, Fig. 4), mostly at the boundaries of the larger Z. The odd nuclei exhibit no definite regularity with respect to the half-lives of the plus-active nuclei.

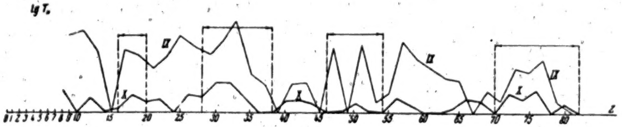

FIG. 4. Curve *IX* -- logarithms of the half-lives of the first plus-active nuclei; *X* -- gamma-ray energy of first unstable $\overline{\beta}$ isotope.

15. *K* electron capture is concentrated principally in the specific zones with the half-life being highest for even nuclei[6].

In addition, let us consider the gamma-radiation curve of the isotope that is one unit mass number smaller than the stable nucleus. This isotope emits a $\overline{\beta}$-particle. The greater the change produced by this emission in the stable nucleus, the higher the energy level reaches by this nucleus (the stronger the excitation of the nucleus) and the greater the energy of the quantum emitted as a result of the β-radiation. The gamma-ray energies of such isotopes for even Z are plotted in curve X of Fig. 4; this curve has minima between $Z = 30$ and $Z = 38$, between $Z = 45$ and $Z = 54$, and between $Z = 78$ and $Z = 82$. If we disregard the very low abundance isotopes, the first zone also contains minima. As far as the odd nuclei are concerned, they have maximum gamma-ray energies in the same regions. The small gamma-ray energy associated with the emission of a β-particle is apparently proof that the nucleus is very close to stable equilibrium. Thus, we obtain the sixteenth feature:

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[6] M. A. Levitskaia and L. P. Rapoport, Dokl. Akad. Naku SSSR 70, 817 (1950)

[5] M. A. Levitskaia, Dokl. Akad. Nauk SSSR 61, 55 (1948)

16. The energy of the gamma-rays accompanying the decay of the first $\overline{\beta}$-active isotope has a minimum in the region of the specific zones.

The seventeenth feature is the following fact reported earlier by myself and by Rapoport[7]:

17. The isomer states are concentrated almost exclusively in the specific zones.

In nuclear transmutations, the specific zones manifest themselves in the $(n, 2n)$ reaction. The emission of two neutrons by capture of a single neutron can naturally be proof of a stronger or weaker neutron bond in the nucleus. The $(n, 2n)$ reaction is observed principally in the first-lightest isotope. In even nuclei having large Z it occurs also in the intermediate nuclei. However, we shall only consider the occurrence of the reaction in the first-lightest isotope. In the case of even nuclei this reaction is not observed in the specific zones. In the case of Ca its occurrence is doubtful, and it does not occur for $Z = 16$ and $Z = 18$. In the second zone it is entirely absent for $Z = 34$, 36, 38, and it is absent for the first stable isotope of $Z = 32$. It is absent in the third zone for $Z = 46$, and is observed for $Z = 50$ only in the last heaviest isotope which is of very low content. It is absent for the first isotope of $Z = 52$, and is entirely absent for $Z = 54$. Above $Z = 56$ the data are generally doubtful; the $(n, 2n)$ reaction has not been observed for the first isotopes of even nuclei, but is always observed for odd nuclei. This again proves the greater stability of the even nuclei, particularly in the specific zones. Thus, we have another feature:

18. The $(n, 2n)$ reaction does not occur for the lightest isotopes of even nuclei in the specific zones.

19. The next feature is the increased stability of the isobars in the specific zones, as reported in reference 8.

In general, the absolute value of binding energy of an isobar increases with Z, but the opposite takes place in the specific zone: the difference between the energies of the isobars with larger and smaller Z starts to decrease at the beginning of the specific zone, and in the center of the zone the isobar with smaller Z has a higher binding energy than the isobar corresponding to the larger Z. This is clearly pronounced in the first and second specific zones, as shown in the Table. Unfortunately, the binding energies are unknown for the remaining specific zones, but the same is expected to hold there although it has not yet been observed.

20. The last feature is the fluctuation in the specific zones of the Z_A numbers and of the lower point of the Bethe-Weizsaecker parabolas of the isobars, as reported by I. Curie[9]. The author of this reference gives a curve of Z_A versus Z and versus A. The value of Z increases almost linearly but fluctuates widely (sinusoidally) in several regions. These regions are the Z intervals 16-21, 29-35, 42-43, 48-52, 61-64, 66-68, 56-58 and 71-81. These regions are the specific zones, regions near unstable Z, or regions containing naturally radioactive isotopes. I. Curie's associates[10] have replaced the single isobar parabola in these regions with two intersecting parabolas to obtain a smoother plot for Z_A. There is no doubt, however, that in the specific zones the fluctuations of Z_A are partly related to the nineteenth feature. The specific zones do not exhibit any unique behavior with respect to such important nuclear quantities as the spin and quadrupole moment. In fact, for odd Z up to $Z = 43$ the maximum spins lie at the upper bounds of the specific zones. Beyond $Z = 43$, the maximum spins are observed at both boundaries of the specific zones. The same can also be

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[7] M. A. Levitskaia and L. P. Rapoport, Dokl. Akad. Nauk SSSR 74, 953 (1950)

[8] M. A. Levitskaia, Tr. VGU (Reports pf Voronezh State University), Phys. Math. Collection, 1954

[9] I. Curie, J. Phys. et Radium 6, 209 (1945)

[10] R. Bouchez, J. Robert and J. Tobailem, J. Phys. et Radium 14, 281 (1953)

Binding Energies of Isobars and Their Differences

Isobar	Binding Energy	Difference	Isobar	Binding Energy	Difference
S^{32}_{16}	271,71	2,23	Cr^{51}_{24}	444,91	—0,85
Si^{32}_{14}	269,48		Ti^{51}_{22}	445,76	
A^{36}_{18}	306,14	—1,5	Fe^{54}_{26}	469,27	—4,...
S^{36}_{16}	307,64		Cr^{54}_{24}	473,...	
Ca^{40}_{20}	340,40	—1,22	Ni^{58}_{28}	500,543	—3,82
A^{40}_{18}	341,62		Fe^{58}_{26}	504,360	
Ti^{46}_{22}	397,32	2,...	Zn^{64}_{30}	553,219	—9,031
Ca^{46}_{20}	(395,...)		Ni^{64}_{23}	562,250	
Ti^{48}_{22}	416,73	3,...	Ge^{70}_{32}	554,31	—10,00
Ca^{48}_{20}	413,...		Zn^{70}_{30}	604,31	
Cr^{50}_{24}	435,30	1,02	Se^{74}_{34}	640,192	—3,325
Ti^{50}_{22}	434,28		Ge^{74}_{32}	643,517	

said concerning the quadrupole moments, as it follows from the Gordy curve (see Fig. 1 in reference 11).

It is next necessary to describe the behavior of the energy in the specific zones. The average binding energy for a single particle hardly means anything in view of the large number of isotopes for even Z. Let us plot the energy of combination of one proton and one neutron, inasmuch as this is made possible by the existing tables of atomic masses.

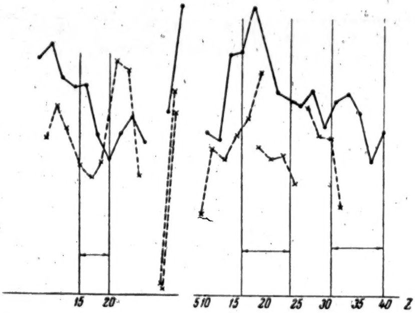

FIG. 5. *a* -- combining energy of one proton plotted as a function of Z; *b* -- combining energy of a single neutron.

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[11] M. A. Levitskaia, Dokl. Akad. Nauk SSSR 74, 37 (1950)

Figure 5*a* shows the energy of combination of a proton as a function of *Z*. A solid line joins the even-proton points and a dotted line joins the odd-proton points (crosses). We see that the energy of combination of the proton drops off in the first specific zone and has a deep minimum at the upper end of the zone for both odd and even protons. Figure 5*b* shows the energy of combination for one neutron. It is difficult to draw solid curves because the energy has several values for the same number of neutrons. The first zone contains the 16th to 26th neutrons, the second zone from the 30th upward. We see again a drop in the energy of combination in the first specific zone and relative maxima at $Z = 18$, 28 and 34.

The value of the combining energy is closely related to the question of nucleon shells and evaluation of these quantities should be the topic of a separate investigation.

All the characteristics indicated in this article are convincing proof that the "specific" zones represent regions of abrupt nuclear changes associated with an increase in the stability of the nuclei, and the magic numbers are apparently a result of these zones.

Translated by J. G. Adashko
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The Quantum Theory of Magnetostriction

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On the basis of the theory of the polar model of a metal in the form developed by Bogoliubov and Tiablikov, starting from the calculation of the magnetic and magnetoelastic interaction of the electrons in the lattice, a step-by-step quantum mechanical theory of magnetostriction of hexagonal monocrystals is developed. The low temperature energy spectrum of the crystal is calculated, as well as the free energy and the temperature dependent of the constants of magnetostriction.

INTRODUCTION

THE quantum theory of ferromagnetism began its development after it had been established by the researches of Frenkel'[1] and later by Heisenberg[2] that the basic property of ferromagnets---the presence of spontaneous magnetization---was explained by electrical exchange forces. The classical theory of magnetic-anisotropic properties of ferromagnets, which is fundamental to the theories of the curves of technical magnetization, is connected with the well-known researches of Akulov[3], who also pointed out that the magnetic anisotropic properties of ferromagnets are determined by the magnetic interaction of electron spins and orbits in the ferromagnetic crystal. The most important magneto-anisotropic properties of ferromagnets are the magnetic-energetic anistropy and magnetostriction. The quantum theory of these properties must be built on the basis of both exchange and magnetic interactions. If we can say that a sufficient number of researches, both Soviet and foreign[4-6], were devoted to the theory of magnetic anisotropy, we must add that the quantum theory of magnetostriction has been only very slightly investigated. Only one research, due to Vonsovskii[7], is devoted to this problem. In this work, a quantum mechanical analysis is given, with the use of the method of energetic centers of gravity, of the phenomenon of magnetostriction in cubic crystals for the region of temperatures close to the Curie point.

The aim of the present work was the step-by-step formation of a quantum mechanical theory of magnetostriction of ferromagnetic monocrystals with hexagonally symmetric lattices at low temperatures. The analysis is made on the basis of the theory of the polar model of a metal in the form developed by the researches of Bogoliubov and Tiablikov[8,9], i.e., in the many electron scheme with the use of the method of approximate second quantization.

MODEL OF THE SYSTEM AND THE INITIAL HAMILTONIAN

We shall consider the crystalline lattice of hexagonal symmetry; let a "ferromagnetic" electron in the ground state be located at each lattice site. We shall assume that the atoms in the lattice are placed sufficiently far apart that the overlap of the orbits of electrons at neighboring sites is not great. We shall assume that the integral of non-orthogonality of the atomic wave functions of the different nodes is small in comparison to unity. In the corresponding scheme of excitation, the wave function of the ground state is completely determined by the occupation number for all sites. It is also assumed that the ground level of the system is separated from the excited levels by an energy gap. We shall consider the introduction of a system of ferromagnetic electrons in this lattice only in the range of low temperatures, where the excitation of the system can be regarded as weak.

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[1] Ia. I. Frenkel', Z. Phys. 49, 31 (1928)

[2] W. Heisenberg, Z. Phys. 49, 619 (1928)

[3] N. S. Akulov, *Ferromagnetism*, Moscow, 1939

[4] S. V. Vonsovskii, J. Exper. Theoret. Phys. USSR 8, 1104 (1938)

[5] J. Van-Vleck, Phys. Rev. 52, 1178 (1937); H. Brooks, Phys. Rev. 58, 909 (1940)

[6] S. V. Tiablikov, J. Exper. Theoret. Phys. USSR 20, 661 (1950)

[7] S. V. Vonsovskii, J. Exper. Theoret. Phys. USSR 10, 762 (1940)

[8] N. N. Bogoliubov and S. V. Tiablikov, J. Exper. Theoret. Phys. USSR 19, 251, 256 (1950)

[9] N. N. Bogoliubov, *Lectures on Quantum Statistics*, Kiev, 1949 (in Ukrainian)

Let the system be located in an external magnetic field, sufficiently strong so that the crystal can be regarded as magnetized almost to saturation. We shall consider both exchange and magnetic interaction. In this case, as also in reference 6, the Hamiltonian of the system in "equivalent" form can be written in terms of the spin operator

$$\tilde{\mathcal{H}} = G_0 - \sum_{f,\alpha} \mu H^\alpha S_f^\alpha - {}^1/_2 \times \sum_{\substack{f_1, f_2\\ \alpha, \beta}} G_{\alpha\beta}(f_1, f_2)\, S_{f_1}^\alpha S_{f_2}^\beta, \tag{1}$$

where $G_{\alpha\beta}(f_1, f_2)$ is the tensor of the electron interaction, which, upon ignoring the magnetic interaction, degenerates into a scalar which represents the usual exchange integral; G_0 is a constant in the sense that it is independent of the spin operators.

The magnetic interaction of the electrons in the ferromagnetic lattice brings about a displacement of the atoms from their equilibrium positions, which brings about a spontaneous deformation of the lattice. We shall consider that these displacements are not large, and that the deformations are homogeneous. Inasmuch as one can consider that $G_{\alpha\beta}(f_1, f_2) = G_{\alpha\beta}(f_1 - f_2)$ in an actual lattice, we expand each of the components of the tensor $G_{\alpha\beta}$ in a series of small displacements relative to the equilibrium position, and restrict ourselves to terms which are linear in the components of the deformation tensor u_{ij}. Then (keeping the previous designation f for the positions of the sites in equilibrium), the equivalent Hamiltonian of the spontaneously deformed lattice can be written in the form

$$\tilde{\mathcal{H}} = G_0 - \sum_{f,\alpha} \mu H \gamma^\alpha S_f^\alpha - {}^1/_2 \times \sum_{\substack{f_1, f_2\\ \alpha, \beta}} G_{\alpha\beta}(f_1, f_2)\, S_{f_1}^\alpha S_{f_2}^\beta - {}^1/_2 \sum_{\substack{f_1, f_2\\ \alpha, \beta}} \sum_{ij} A_{ij}^{\alpha\beta}(f_1, f_2)\, S_{f_1}^\alpha S_{f_2}^\beta u_{ij} \tag{2}$$

or

$$\tilde{\mathcal{H}} = G - {}^1/_2 \sum_{\substack{f_1, f_2\\ \alpha, \beta}} D_{\alpha\beta}(f_1, f_2)\, S_{f_1}^\alpha S_{f_2}^\beta - \sum_{f,\alpha} \mu H \gamma^\alpha S_f^\alpha, \quad D_{\alpha\beta}(f_1, f_2) = G_{\alpha\beta}(f_1, f_2) + \sum_{ij} u_{ij} A_{ij}^{\alpha\beta}(f_1, f_2), \tag{3}$$

where $A_{ij}^{\alpha\beta}(f_1, f_2)$ is the tensor of magnetic interaction, γ^α is the direction cosine of the magnetic field. (Inasmuch as the tensor $G_{\alpha\beta}$ describes the magnetic interaction in generalized form, the evident form of the dependence of $A_{ij}^{\alpha\beta}$ on the components $G_{\alpha\beta}$ and their derivatives is not brought out here, and will not be made use of in what follows.) In this manner, the Hamiltonian of the system in the form (2) will appear as the initial Hamiltonian of our problem.

The calculation of the terms of magnetic and magneto-elastic interaction permits the removal of exchange degeneracy even in the zeroth approximation. The parametric character of the dependence of the Hamiltonian (2) on the components of the deformation tensor makes possible the use of the general scheme of the method of finding the ground level and the energy spectrum that is described in the book of Bogoliubov[9]. For this purpose, we express the Hamiltonian (3) by Fermi operators $a_{f\nu}$, making use of the well-known relations

$$S_f^x = a_{f,-1/2}^+ a_{f,1/2} + a_{f,1/2}^+ a_{f,-1/2},$$
$$S_f^y = i(a_{f,1/2}^+ a_{f,-1/2} - a_{f,-1/2}^+ a_{f,1/2}),$$
$$S_f^z = a_{f,-1/2}^+ a_{f,-1/2} - a_{f,1/2}^+ a_{f,1/2}, \tag{4}$$

with the condition

$$a_{f,-1/2}^+ a_{f,-1/2} + a_{f,1/2}^+ a_{f,1/2} = 1. \tag{5}$$

The Hamiltonian takes the form

$$\tilde{\mathcal{H}} = - {}^1/_2 \sum B(f_1, f_2, \nu_1, \nu_2, \nu_1', \nu_2')\, a_{f_1\nu_1}^+ a_{f_2\nu_2}^+ a_{f_2\nu_2'} a_{f_1\nu_1'} + \sum A(f, \nu_1, \nu_1')\, a_{f\nu_1}^+ a_{f\nu_1'} \tag{6}$$

(for the present, we omit the constant term G_0). The quantities $B(f_1, f_2, \nu_1, \nu_2, \nu_1', \nu_2')$ and $A(f, \nu_1, \nu_1')$ will be completely determined as known functions of the components of the tensors $G_{\alpha\beta}$, $A_{ij}^{\alpha\beta}$ u_{ij} and the field H.

DETERMINATION OF THE GROUND LEVEL AND THE ENERGY SPECTRUM

The ground level of the system is found by a quasi-classical method, replacing the operators $a_{f\nu}$ by the c-numbers $\theta_0(f, \nu)$ which are subject to the condition

$$\sum \theta_0^{*}(f, \nu)\,\theta_0(f, \nu) = 1 \tag{7}$$

and which satisfy the equations

$$-\sum B(f_1, f_2, \nu_1, \nu_2, \nu_1', \nu_2') \tag{8}$$

$$\times\ \theta_0^{*}(f_2, \nu_2)\,\theta_0(f_2, \nu_2')\,\theta_0(f_1, \nu_1')$$

$$+\sum A(f_1, \nu_1, \nu_2)\,\theta_0(f_1, \nu_2) = \lambda_0(f_1)\,\theta_0(f_1 \nu_1).$$

We transform Eq. (6) to the operators $a_{f\omega}$ with the help of the function $\theta_\omega(f, \nu)$:

$$a_{f\nu} = \sum_{\omega} \theta_\omega(f, \nu)\, a_{f\omega} \qquad (\omega = 0.1),$$

where the $\theta_\omega(f, \nu)$ are orthogonal to $\theta_0(f, \nu)$ and satisfy the equations

$$-\sum B(f_1, f_2, \nu_1, \nu_2, \nu_1', \nu_2') \tag{9}$$

$$\times\ \theta_0^{*}(f_2, \nu_2)\,\theta_0(f_2, \nu_2')\,\theta_\omega(f_1, \nu_1')$$

$$+\sum A(f_1, \nu_1, \nu_2)\,\theta_\omega(f_1, \nu_2) = \lambda_\omega(f_1)\,\theta_\omega(f_1, \nu_1).$$

Introducing the operators $b_{f\omega}$ which obey the Bose statistics aproximately:

$$b_{f\omega} = \overset{+}{a}_{f0} a_{f\omega},\ \overset{+}{b}_{f\omega} \tag{10}$$

$$= \overset{+}{a}_{f\omega} a_{f0},\ n_{f\omega} = \overset{+}{b}_{f\omega} b_{f\omega} \ (\omega = 1),$$

The Hamiltonian (6) can be reduced, for the case of weak excitations, to a quadratic form relative to the operators $b_{f\omega}$:

$$\tilde{\mathscr{H}} = E_0 + \sum \{\lambda_\omega(f) - \lambda_0(f)\}\, \overset{+}{b}_{f\omega} b_{f\omega} \tag{11}$$

$$+ \sum Q(f_1, f_2, \omega_1, \omega_2)\, \overset{+}{b}_{f_1\omega_1} b_{f_2\omega_2}$$

$$+ {}^1\!/_2 \sum P^{*}(f_1, f_2, \omega_1, \omega_2)\, b_{f_1\omega_1} b_{f_2\omega_2}$$

$$+ {}^1\!/_2 \sum P(f_1, f_2, \omega_1, \omega_2)\, \overset{+}{b}_{f_1\omega_1} \overset{+}{b}_{f_2\omega_2},$$

where

$$Q(f_1, f_2, \omega_1, \omega_2) \tag{12}$$

$$= \sum B(f_1, f_2, \nu_1, \nu_2, \nu_1', \nu_2')$$

$$\times\ \theta_{\omega_1}^{*}(f_1, \nu_1)\,\theta_0^{*}(f_2, \nu_2)\,\theta_{\omega_2}(f_2, \nu_2')\,\theta_0(f_1, \nu_1'),$$

$$P(f_1, f_2, \omega_1, \omega_2)$$

$$= \sum B(f_1, f_2, \nu_1, \nu_2, \nu_1', \nu_2')$$

$$\times\ \theta_{\omega_1}^{*}(f_1, \nu_1)\,\theta_{\omega_2}^{*}(f_2, \nu_2)\,\theta_0(f_2, \nu_2')\,\theta_0(f_1, \nu_1').$$

The problem of finding the energy spectrum reduces to the reduction of the Hamiltonian (11) to diagonal form. For this, we transform Eq. (11) according to the formulas

$$b_{f1} = \sum_k \{u_{kf}\xi_k + v_{kf}^{*}\overset{+}{\xi}_k\}, \tag{13}$$

$$\overset{+}{b}_{f1} = \sum \{u_{kf}^{*}\overset{+}{\xi}_k + v_{kf}\xi_k\},$$

where the ξ_k are Bose operators and u_{kf} and v_{kf} are the eigenfunctions of the equations

$$E_k u_{kf_1} = \sum_{f_2} P(f_1, f_2)\, v_{kf_2} \tag{14}$$

$$+ \sum_{f_2} Q(f_1, f_2)\, u_{kf_2} + \Lambda_{f_1} u_{kf_1},$$

$$-E_k v_{kf_1} = \sum_{f_2} P^{*}(f_1, f_2)\, u_{kf_2}$$

$$+ \sum_{f_2} Q^{*}(f_1, f_2)\, v_{kf_2} + \Lambda_{f_1} v_{kf_1}$$

$$(\Lambda_f = \lambda_1(f) - \lambda_0(f)),$$

and satisfy the condition

$$\sum_f (u_{kf} u_{k'f}^{*} - v_{kf} v_{k'f}^{*}) = \delta(k, k') \tag{15}$$

[E_k is the eigenvalue of the system (14)]. Making use of Eq. (13), setting up pair products of the operators $b_{f\omega}$ and substituting in Eq. (11), we obtain, keeping in mind the properties of ξ_k, v_{kf}, u_{kf}

$$\tilde{\mathscr{H}} = E_0 - \sum E_k v_{kf}^{*} v_{kf} + \sum E_k \overset{+}{\xi}_k \xi_k. \tag{16}$$

The quantity $-\Sigma E_k v_{kf}^{*} v_{kf}$ can be regarded as a correction to the energy level E_0, for which reason we denote it as ΔE_0. The quantity E_k is some function of the wave number which characterizes the spectrum of elementary excitations. The eigenvalues of the operator $\xi_k^+ \xi_k = \hat{N}_k$ are known; they are the integers 0, 1, 2, ...

Thus,

$$\mathscr{H} = E_0 + \Delta E_0 + \sum E_k N_k. \tag{17}$$

The energy of the system with the Hamiltonian in the form (17) can be interpreted as the energy of an ideal gas of quasi-particles (elementary excitations), which obeys the Bose statistics; the quantity E_k will characterize the energy spectrum of the quasi-particles. The energy levels of the system will be determined by the collection of occupation numbers of the quasi-particles in the states E_k. In order to establish the dependence of the quantities ΔE_0 and E_k (which are of interest to us) on the characteristics of the system, we make use of Eqs. (14), the solutions to which we shall seek in the form

$$u_{kf} = u_k e^{ifk}, \quad v_{kf} = v_k e^{ifk}. \tag{18}$$

Substituting Eq. (18) in Eq. (14) and introducing the notation

$$P(k) = \sum_{f_2} P(f_1 - f_2)\, e^{i(f_1-f_2)k}, \tag{19}$$

$$Q(k) = \sum_{f_2} Q(f_1 - f_2)\, e^{i(f_1-f_2)k},$$

we get

$$[(E_k - \Lambda) - Q(k)]\, u_k - P(k)\, v_k = 0, \tag{20}$$

$$P^*(k)\, u_k - [(E_k + \Lambda) + Q^*(k)]\, v_k = 0.$$

From the conditions of solvability of these equations, an expression is immediately obtained for $E(k)$, if we keep in mind that $Q(k)$ is a real quantity,

$$E_k = \{(Q(k) - \Lambda)^2 - |P(k)|^2\}^{1/2}. \tag{21}$$

In order to find ΔE_0 we must calculate $v_{kf} v^*_{kf}$; by virtue of Eq. (18) it is sufficient to find v_k^2. From Eq. (20), eliminating u_k and making use of the condition (15), we get

$$\Delta E_0 = -\frac{1}{2} \sum \frac{|P(k)|^2}{E_k + Q(k) + \Lambda} \tag{22}$$

To bypass the difficulties of solution of the system of Eqs. (8) and (9), we take advantage of the procedure set forth by Tiablikov[6]. In finding the the ground level and the minimizing form of the function $\theta_0(f, \nu)$ we can replace the components of the spin operator S_f^α by components of ordinary vectors σ_f^α. We put the relations which connect the components σ_f with the c-numbers $\theta_0(f, \nu)$, by analogy with Eq. (4), in the form

$$\sigma_f^x = \theta_0^*(f, -1/2)\, \theta_0(f, 1/2) \tag{23}$$

$$+ \theta_0^*(f, 1/2)\, \theta_0(f, -1/2),$$

$$\sigma_f^y = i\, \{\theta_0^*(f, 1/2)$$

$$\theta_0(f, -1/2) - \theta_0^*(f, -1/2)\, \theta_0(f, 1/2),$$

$$\sigma_f^z = \theta_0^*(f, -1/2)\, \theta_0(f, -1/2)$$

$$- \theta_0^*(f, 1/2)\, \theta_0(f, 1/2).$$

Instead of Eq. (7) we will have the condition

$$\sum_\alpha (\sigma_f^\alpha)^2 = 1 \quad (\alpha = x, y, z). \tag{24}$$

Keeping in mind the well-defined connection between σ_f^α and $\theta_0(f, \nu)$, we consider the quadratic form

$$E = -1/2 \sum D_{\alpha\beta}(f_1, f_2, u_{ij})\, \sigma_{f_1}^\alpha \sigma_{f_2}^\beta \tag{25}$$

$$- \sum \mu H \gamma^\alpha \sigma_f^\alpha$$

and seek its minimum under the added condition (24). Consequently, the minimizing form of the value of the components σ_f^α can be determined from the system of $3N$ equations

$$-\sum_{f_2, \beta} D_{\alpha\beta}(f_1, f_2, u_{ij})\, \sigma_{f_2}^\beta - \lambda(f_1)\, \sigma_f^\alpha = \mu H \gamma^\alpha. \tag{26}$$

We now find the expression for E_0. For this purpose, we multiply each of the Eqs. (26) by the corresponding σ_f^α and add. As a result, we obtain

$$E_0 = 1/2 \sum D_{\alpha\beta}(f_1, f_2, u_{ij})\, \sigma_{f_1}^\alpha \sigma_{f_2}^\beta + \sum \lambda(f_1). \tag{27}$$

The equation for the calculation of $\theta_0(f, \nu)$ in terms of σ_f^α can be written in the form

$$\lambda_0(f_1)\, \theta_0(f_1, \nu) = \sum_\alpha \lambda(f_1)\, \sigma_{f_1}^\alpha\, [\partial \sigma_{f_1}^\alpha / \partial \theta_0^*(f_1, \nu)]. \tag{28}$$

We note that Eq. (9) for determining $\theta_1(f, \nu)$ is obtained from Eq. (8) by the formal substitution of $\theta_1(f_1, \nu_1)$ for $\theta_0(f_1, \nu_1)$ (here it is obvious that $\lambda_0 \to \lambda_1$). In the same way we get from Eq. (28) the equation for $\theta_1(f_1, \nu_1)$.

$$\lambda_1(f_1)\, \theta_1(f_1, \nu_1) \tag{29}$$

$$= \sum_\alpha \lambda(f_1)\, \sigma_{f_1}^\alpha\, [\partial \sigma_{f_1}^\alpha / \partial \theta_0^*(f_1, \nu_1)]_{\theta_0(f_1, \nu_1) \to \theta_1(f_1, \nu_1)}.$$

We solve Eqs. (28) and (29) by making use of the orthogonality of the functions $\theta_0(f, \nu)$, $\theta_1(f, \nu)$; we obtain

$$\theta_0(f, -{}^1/_2) = \sqrt{(1+\sigma_f^z)/2},\ \theta_0(f, {}^1/_2) = e^{i\varphi_f}\sqrt{(1-\sigma_f^z)/2}, \tag{30}$$

$$\theta_1(f, -{}^1/_2) = \sqrt{(1-\sigma_f^z)/2},\ \theta_1(f, {}^1/_2) = e^{i(\varphi_f+\pi)}\sqrt{1+\sigma_f^z)/2}, \tag{31}$$

$$\lambda_1(f_1) = -\lambda_0(f_1) = \lambda(H).$$

It was shown earlier that the external magnetic field is sufficiently strong; moreover, we assume that because of the nearness of the system to the state of magnetic saturation of the spin of all sites one can consider the spins parallel and therefore the components σ_f^α are practically independent of the site number f.

We introduce sums over the lattice

$$\overline{G}_{\alpha\beta} = \sum_{f_2} G_{\alpha\beta}(f_1, f_2),\ \overline{A}_{ij}^{\alpha\beta} = \sum_{f_2} \overline{A}_{ij}^{\alpha\beta}(f_1, f_2). \tag{32}$$

For a hexagonal lattice (when the principal axis coincides with the axis $0Z$) the following assumptions can be made relative to the components of the tensors $G^{\alpha\beta}$ and $A_{ij}^{\alpha\beta}$:

$$\overline{G}_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta; \tag{33}$$

$$\overline{G}_{11} = \overline{G}_{22} = G_0;\quad \overline{G}_{33} = G_0^a > G_0;$$

$$\overline{A}_{ij}^{\alpha\beta} = \overline{A}_{ji}^{\alpha\beta} = \overline{A}_{ij}^{\beta\alpha};\quad \overline{A}_{ij}^{\alpha\beta} = \overline{A}_{\alpha\beta}^{ij}, \tag{34}$$

$$A_{ij}^{\alpha\beta} \neq 0 \begin{cases} \text{when } (\alpha\beta) \neq (ij) \text{ for } i=j \text{ and simultane-} \\ \text{ously } \alpha = \beta; \\ \text{when } (\alpha\beta) = (ij) \text{ for } i \neq j,\ \alpha \neq \beta; \end{cases}$$

the other components vanish.

Introducing the abbreviating notation:

$$-\frac{G_0+\lambda}{\mu H} = x;\quad G_0^a - G_0 = \Delta;\ \frac{\Delta}{\mu H} = \eta; \tag{35a}$$

$$A_1^1/\mu H = \varepsilon_1,\ A_2^2/\mu H = \varepsilon_2,\ A_3^3/\mu H = \varepsilon_3;\ A_2^1/\mu H = \varepsilon_1^0,\ A_2^3/\mu H = \varepsilon_2^0;$$

$$A_1^3/\mu H = \varepsilon_3^0, \tag{35b}$$

where

$$A_\alpha^\beta = A_\beta^\alpha = \sum_{ij} u_{ij} A_{ij}^{\alpha\beta}, \tag{35c}$$

we rewrite Eq. (26), taking into account the assumptions on the independence of σ_f^α on f:

$$(\varepsilon_1 - x)\sigma_1 + \varepsilon_1^0\sigma_2 + \varepsilon_3^0\sigma_3 = -\gamma_1, \tag{36}$$

$$\varepsilon_1^0\sigma_1 + (\varepsilon_2 - x)\sigma_2 + \varepsilon_2^0\sigma_3 = -\gamma_2,$$

$$\varepsilon_3^0\sigma_1 + \varepsilon_2^0\sigma_2 + (\varepsilon_3 + \eta - x)\sigma_3 = -\gamma_3.$$

According to Eq. (24), the σ_i are connected by the condition

$$\sum_i \sigma_i^2 = 1. \tag{37}$$

Because of the assumption on the strong fields, the magnitudes of η, ϵ_i, ϵ_i^0 can be considered small, and Eqs. (36) can be solved approximately under the condition (37). As a result, we obtain relatively simple expressions for σ_i and λ:

$$\sigma_1 = \frac{1}{x}\left\{\gamma_1 + \frac{1}{x}\left[\gamma_1\varepsilon_1 + \gamma_2\varepsilon_1^0 + \gamma_3\varepsilon_3^0\right]\right\}, \tag{38}$$

$$\sigma_2 = \frac{1}{x}\left\{\gamma_2 + \frac{1}{x}\left[\gamma_1\varepsilon_1^0 + \gamma_2\varepsilon_2 + \gamma_3\varepsilon_2^0\right]\right\},$$

$$\sigma_3 = \frac{1}{x-\eta}\left\{\gamma_3 + \frac{1}{x-\eta}\left[\gamma_1\varepsilon_3^0 + \gamma_2\varepsilon_2^0 + \gamma_3\varepsilon_3\right]\right\};$$

$$\lambda = -G_0 - \mu H - \Delta\gamma_3^2 - \xi(\gamma_i, \gamma_j, u_j), \tag{39}$$

where

$$\begin{aligned} \xi(\gamma_i, \gamma_j, u_{ij}) = {} & (u_{11}\overline{A}_{11}^{11} + u_{22}\overline{A}_{22}^{11} + u_{33}\overline{A}_{33}^{11})\gamma_1^2 \\ & + (u_{11}\overline{A}_{11}^{22} + u_{22}\overline{A}_{22}^{22} + u_{33}\overline{A}_{33}^{22})\gamma_2^2 \\ & + (u_{11}\overline{A}_{11}^{33} + u_{22}\overline{A}_{22}^{33} + u_{33}\overline{A}_{33}^{33})\gamma_3^2 + 2u_{12}\overline{A}_{12}^{12}\gamma_1\gamma_2 \\ & + 2u_{13}\overline{A}_{13}^{13}\gamma_1\gamma_3 + 2u_{23}\overline{A}_{23}^{23}\gamma_2\gamma_3. \end{aligned} \tag{40}$$

Substituting in the formulas for $P(f_1, f_2)$ and $Q(f_1, f_2)$ expressions for $B(f_1, f_2, \nu_1, \nu_2, \nu_1', \nu_2')$,

$\theta_0(f, \nu)$ and $\theta_1(f, \nu)$, we obtain

$$\begin{aligned}-Q(f_1,f_2) &= \frac{1}{2}[D_{11}(f_1, f_2)+D_{22}(f_1, f_2)](1+\sigma_3^2)\\ &+D_{33}(f_1, f_2)(1-\sigma_3^2)-D_{12}(f_1, f_2)(1-\sigma_3^2)\sin 2\varphi\\ &-\frac{1}{2}[D_{11}(f_1, f_2)-D_{22}(f_1, f_2)](1-\sigma_3^2)\cos 2\varphi\\ &-2D_{13}(f_1, f_2)\sigma_3\sqrt{1-\sigma_3^2}\sin\varphi-2D_{23}(f_1, f_2)\sigma_3\sqrt{1-\sigma_3^2}\cos\varphi;\end{aligned} \tag{41}$$

$$\begin{aligned}-P(f_1,f_2) &= [D_{11}(f_1, f_2)-D_{22}(f_1, f_2)]\left\{\frac{1}{2}(1-\sigma_3^2)\cos 2\varphi-\sigma_3\sin 2\varphi\right\}\\ &-2iD_{12}(f_1, f_2)\left\{\frac{1}{2}(1+\sigma_3^2)\sin 2\varphi-\sigma_3\cos 2\varphi\right\}\\ &+\frac{1}{2}[2D_{33}(f_1,f_2)-D_{22}(f_1,f_2)-D_{11}(f_1,f_2)](1-\sigma_3^2)\\ &+[D_{13}(f_1, f_2)-iD_{23}(f_1,f_2)]e^{i\varphi}(1-\sigma_3)\sqrt{1-\sigma_3^2}\\ &-[D_{13}(f_1,f_2)+iD_{23}(f_1, f_2)]e^{-i\varphi}(1+\sigma_3)\sqrt{1-\sigma_3^2},\end{aligned}$$

where

$$\cos\varphi=\frac{\sigma_1}{\sqrt{1-\sigma_3^2}}, \qquad \sin\varphi=\frac{\sigma_2}{\sqrt{1-\sigma_3^2}}, \qquad \varphi=\operatorname{arctg}\frac{\sigma_2}{\sigma_1}. \tag{42}$$

We now have arranged all the necessary data for finding the quantities E_0, ΔE_0 and E_k of interest to us.

Making use of Eqs. (27) and (33) we get for E_0

$$E_0=-\frac{N}{2}\left[G_0+\frac{1}{2}\Delta\gamma_3^2+\frac{1}{2}\xi(\gamma_i, \gamma_j, u_{ij})+2\mu H\right] \tag{44}$$

We introduce the Fourier decomposition

$$\begin{aligned}D_{\alpha\beta}(k) &= \sum_{(f_2-f_1)\neq 0} D(f_2-f_1)e^{-i(f_2-f_1)k}\\ &= G_{\alpha\beta}(k)+\sum_{ij}u_{ij}A_{ij}^{\alpha\beta}(k).\end{aligned} \tag{45}$$

Inasmuch as we have made assumptions about low temperatures, we must limit ourselves to the approximation of small wave numbers

$$G_{\alpha\beta}(k)=\overline{G_{\alpha\beta}}-\frac{1}{2}\overline{G_{\alpha\beta}f^2}\,k^2+O_G(k^4), \tag{46}$$

$$A_{ij}^{\alpha\beta}(k)=\overline{A_{ij}^{\alpha\beta}}-\frac{1}{2}\overline{A_{ij}^{\alpha\beta}f^2}\,k^2+O_A(k^4).$$

In such an assumption, it is of course understood that we limit ourselves to the approximation of nearest neighbors and make a definite averaging over the angles in the space of the wave numbers.

For maximum simplification of the tedious calculations, we shall use approximate forms in place of Eqs. (21) and (22):

$$E_k=Q(k)-2\lambda, \tag{47}$$

$$\Delta E_0=-\frac{1}{4}\sum\frac{|P(k)|^2}{E_k}. \tag{48}$$

Multiplying on the right and the left of the expression for $Q(f_1, f_2)$ in Eq. (41) by $e^{-i(fk)}$ and summing over f, we get $Q(k)$. Expanding the components

$$D_{\alpha\beta}(k)=G_{\alpha\beta}(k)+\sum u_{ij}A_{ij}^{\alpha\beta}(k)$$

in a series, in accord with Eq. (46), and substituting in $Q(k)$, and also substituting for the σ_i their values from Eq. (38) [taking into account Eq. (35)], we get

$$E_k=2\mu H-\Delta(1-3\gamma_3^2)+\xi(\gamma_i, \gamma_j, u_{ij})+\beta k^2. \tag{49}$$

The expression for β has a rather formidable appearance, but the first term is the greatest:

$$\beta = \frac{1}{2}\left[\overline{G_0^a f^2} + \overline{G_0 f^2}\right] + \ldots \tag{50}$$

It is not difficult to note that this quantity is proportional to the exchange integral. To find ΔE_0, in accordance with Eq. (48), we must find $|P(k)|^2$. The quantity $P(k)$ can be found in the same way as $Q(k)$, from Eq. (42) for $P(f_1, f_2)$. Inasmuch as $P(k)$ is a complex quantity, we can write it in the form

$$P(k) = P_r(k) + iP_i(k). \tag{51}$$

In this same approximation of Eq. (46) we get

$$|P(k)|^2 = a_0 - a_2 k^2 + a_4 k^4, \tag{52}$$

where the largest term, which we need, is

$$a_0 = \overline{P}^2_{r0} + \overline{P}^2_{i0}. \tag{53}$$

Denoting a certain limiting value of k by n (n = a number of the order of unity), we get, approximately, for the case of moderately strong fields (omitting terms which do not depend on the field):

$$\Delta E_0 = \frac{a_0 V}{16\pi\beta^{3/2}} \sqrt{2\mu H - \Delta(1 - 3\gamma_3^2) + \xi(\gamma_i, \gamma_j, u_{ij})}. \tag{54}$$

We note that term which corresponds to ΔE_0 in the expression for the magnetization (M_H), and which is obtained after differentiation according to H, not being dependent on temperature, keeps a finite value even at absolute zero and disappears only in infinitely large fields. This conclusion is still further confirmed by the results obtained in the researches of references 6 and 10. However, the expression for M_H is obtained much more precisely by virtue of the calculation of the magneto-elastic interaction (within the framework of the assumption on a single ferromagnetic electron per atom)*.

An estimate of the quantity a_0 shows that $a_0 \approx \Delta^2$ and, inasmuch as Δ is of the order of the first constant of magnetic anisotropy (relative to one atom), i.e., of the order of 10^{-16} - 10^{-17} erg or less, in the computation of the free energy and magnetostriction, we will not consider the correction to the ground level, since its effect on the final results will be very slight in the accepted approximation.

THE FREE ENERGY AND THE TEMPERATURE DEPENDENCE OF THE CONSTANTS OF MAGNETOSTRICTION

All quantities which characterize the energy of a spontaneously deformed crystal have now been obtained. In order to go over to macroscopic quantities, we find the sum-over-states

$$Z = \mathrm{Spur}\,\{e^{-\mathcal{H}/\vartheta}\} \qquad (\vartheta = kT). \tag{55}$$

In our case the Hamiltonian $\mathcal{H}$ has the form of Eq. (17); therefore, we obtain the following expression for the free energy in the usual way:

$$\Psi = E_0 + \Delta E_0 + \vartheta\Sigma \ln(1 - e^{-E_k/\vartheta}). \tag{56}$$

Making use of Eqs. (44) and (49) for E_0 and E_k, we obtain, approximately,

$$\begin{aligned}\Psi = -\frac{1}{2}\frac{N}{V}[G_0 + \Delta\gamma^2{}_3 &+ \xi(\gamma_i, \gamma_j, u_{ij}) + 2\mu H] \\ &- \frac{\vartheta}{2\pi^2}\frac{\sqrt{\pi}}{4}\left(\frac{\vartheta}{\beta}\right)^{3/2} \exp\left\{-\frac{2\mu H - \Delta}{\vartheta}\right\} \\ &\times \exp\left\{-\frac{3\Delta\gamma_3^2 + \xi(\gamma_i, \gamma_j, u_{ij})}{\vartheta}\right\}\end{aligned} \tag{57}$$

We are primarily interested in the character of the temperature dependence of the anisotropic and magneto-elastic terms of the free energy; therefore, we have specifically isolated in the second term of Eq. (57) the factor which contains the direction cosines of the vector of the magnetic field (at saturation, the direction of the field coincides with the direction of the magnetization).

At temperatures differing only slightly from absolute zero, the first term in Eq. (57) is the better approximation for the free energy. At high temperatures, the second term of Eq. (57) also begins to play a role. The exponential factor

$$\exp\{-[3\Delta\gamma_3^2 + \xi(\gamma_i, \gamma_j, u_{ij})]/\vartheta\} \tag{58}$$

at small values of the exponent can be expanded in a series. Inasmuch as the quantity $\xi(\gamma_i, \gamma_j, u_{ij})$

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* The problem of the magnetization of ferromagnetics in connection with a calculation of the magnetic interaction, touched upon here very superficially, requires special consideration and will be treated in another article.

[10] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940)

is of the order of the magneto-elastic energy per atom, it is 3-4 orders of magnitude smaller than Δ. Therefore, the possibility of decomposition is determined by the condition

$$\vartheta > \Delta. \tag{59}$$

This condition determines the lower temperature limit of applicability of such a decomposition; carrying out the latter, we get for the free energy of anisotropy

$$\Psi_a = - \left\{ \frac{N\Delta}{2V} - \frac{3\sqrt{\pi}\Delta}{8\pi^2} \left(\frac{\vartheta}{\beta}\right)^{3/2} \exp\left[- \frac{2\mu H - \Delta}{\vartheta} \right] \right\} \gamma_3^2 \tag{60}$$

and for the free magneto-elastic energy

$$\Psi_{\text{м.у}} = - \left\{ \frac{N}{2V} - \frac{\sqrt{\pi}}{8\pi^2} \left(\frac{\vartheta}{\beta}\right)^{3/2} \times \exp\left[- \frac{2\mu H - \Delta}{\vartheta} \right] \right\} \xi(\gamma_i, \gamma_j, u_{ij}) = f(\vartheta, H)\, \xi(\gamma_i, \gamma_j, u_{ij}). \tag{61}$$

It is evident by direct comparison that Eq. (60) for the free energy of anisotropy virtually coincides with the expression obtained earlier by Tiablikov[6].

From the properties of the tensor $A_{ij}^{\alpha\beta}(f_1, f_2)$ for the hexagonal lattice, it follows that

$$\overline{A_{11}^{11}} = \overline{A_{22}^{22}},\ \overline{A_{11}^{22}} = \overline{A_{22}^{11}},\ \overline{A_{33}^{11}} = \overline{A_{11}^{33}} = \overline{A_{22}^{33}} = \overline{A_{33}^{22}},\ \overline{A_{23}^{23}} = \overline{A_{13}^{13}},\ \overline{A_{12}^{12}} = 2\,(\overline{A_{11}^{11}} - \overline{A_{22}^{22}}). \tag{62}$$

Inserting these relations in Eq. (61), we obtain an expression, from whose consideration it is evident that it is analogous, in the character of its dependence on γ_i and u_{ij}, to the classical expression for the magneto-elastic energy; hence, the quantities

$$\frac{\overline{A_{ij}^{\alpha\beta}}\, N}{2V} f(\vartheta, H) \tag{63}$$

will play the role of magneto-elastic coefficients.

Up to the present time the dependence of the energetic quantities with which we have to deal on the u_{ij} has been considered parametrically. In order to obtain clear expressions for the magnetostrictive constants, it is necessary to bring into consideration the elastic energy and to determine the equilibrium value of the components of the deformation tensor from the condition of minimum free energy. This is done in precisely the same way as in classical theory. A s a result we obtain for the magnetostrictive constants the expression

$$\varkappa_i = \varkappa_i^0 \left\{ 1 - \frac{V}{4N} \left(\frac{\vartheta}{\pi\beta}\right)^{3/2} \times \exp\left[- \frac{2\mu H - \Delta}{\vartheta} \right] \right\},\quad i = 1, 2, \ldots, 5, \tag{64}$$

where, for example,

$$\varkappa_1^0 = \frac{N}{V}\, [c_1 (2\overline{A_{11}^{33}} - \overline{A_{11}^{11}} - \overline{A_{11}^{22}}) - c_2 \overline{A_{33}^{33}}], \tag{65}$$

$$\varkappa_2^0 = \frac{N}{V}\, [c_3 \overline{A_{11}^{22}} - c_4 \overline{A_{11}^{11}} - c_5 \overline{A_{33}^{33}}].$$

The coefficients c_i depend on the elastic constants of the crystalline lattice.

The quantities $A_{ij}^{\alpha\beta}$ in the complete theory depend on the matrix elements of the operators of magneto-elastic interaction, and are computed with the help of the corresponding wave functions. It is therefore natural that the resultant constants of magnetostriction can be either positive or negative. This circumstance is significant, since the classical theory accounts only for constants of positive sign. This special feature of the quantum theory of magnetostriction was pointed out earlier by Vonsovskii[7]. The dependence of the constants of magnetostriction on the magnitude of the magnetic field (which stems from the theory) is a new and significant result. We note that the necessity of such a dependence was shown by Akulov[3].

The low temperature limit of applicabliity of the resultant formulas for the free energy and the constants of magnetostriction was defined by the inequality (59) above. The upper limit is evidently determined by the basic physical assumption of the theory---consideration of only weakly interacting systems. Referring to this assumption, we throw away terms of third and fourth order relative to the operators $a_{f\omega}$ in the Hamiltonian in the operators $a_{f\omega}$. Therefore, the upper temperature limit can be fixed from an estimate of the energy contribution of the discarded terms and comparison of them with the energy contribution of those that remain. The corresponding calculation gives

$$(\vartheta / J)^{3/2} \ll 1, \tag{66}$$

where J is the exchange integral. If, for example, accuracy within 20% is desired from (66), it follows that the expression for the free energy of a ferromagnetic monocrystal (57) and the formulas obtained from it for the temperature dependence of the constants of magnetostriction are correct to the temperature of liquid hydrogen.

CONCLUSIONS

1. A systematic quantum-mechanical consideration of a system of electrons in a ferromagnetic monocrystal gives the possibility of explaining the phenomenon of magnetostriction and once more confirms that the phenomenon of magnetostriction is essentially connected to the magnetic interaction of the electrons.

2. The theory gives the temperature dependence of the constants of magnetostriction for the hexagonal crystals.

3. The constants of magnetostriction must display a dependence on the magnetic field. In connection with this fact, representations used in quantum theory involving the independence of the magnetic constants on the field require more accurate definition.

4. A definite analogy exists between the temperature dependence of the constants of magnetostriction and magnetic anisotropy, as is evident from the comparison of Eq. (60) with Eq. (61). This analogy is connected with the character of the energetic spectrum of ferromagnetics at low temperatures.

The proposed theory is clearly still incomplete. and needs, in the future, refinement and development. Its deficiencies are connected, in the first place, with the assumed model of the ferromagnetic and, in the second place, with the approximations and assumptions made in the course of the calculations. In particular, it ought to be relieved of the assumption that there must be only one ferromagnetic electron at a site; the representation of the magnetic interaction in generalized tensor form, for all its advantages of generality, still bears a phenomenological character; other types of interaction of electrons in the lattice were not studied; the case of weak fields, which evidently presents definite interest, was not considered. However, the initial aspects of the theory permit us, without any difficulties in principle, to take into account many of these factors.

In conclusion, we note that the method used in the present work for consideration of the magnetostriction of hexagonal crystals is also applicable to other problems of the quantum theory of ferromagnetism and, in particular, it can be extended to cubical crystals and to much more complicated systems (alloys, antiferromagnets). These problems, which have an independent interest, will be considered in subsequent researches.

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The Processes of Production of Heavy Mesons and V_1^0-Particles*

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Production processes of V_1^0-particles and heavy mesons are considered from a phenomenological point of view. A mechanism for associated production of V-particles and heavy mesons is discussed. The possibility is estimated for relatively stable systems to exist, composed of nucleons and V-particles. Detailed consideration is given to the difficulties connected with the existence of particles which are produced in abundance but have a long lifetime.

The conclusions which are reached may help in forming working hypotheses for the interpretation of experimental data and for the design of experiments on the production of the new particles.

1. NUCLEONIC CHARGE

IN considering the problem of the production of V_1^0-particles and heavy mesons, it is convenient to use the concept of "nucleonic charge". According to Zel'dovich[3], a nucleon has nucleonic charge $+\gamma$, an antinucleon $-\gamma$, and the pion, muon, electron, photon and neutrino have nucleonic charge zero. The total nucleonic charge is conserved in nuclear reactions. This principle expresses the fact that nuclear matter is stable. In particular, it states that a nucleon cannot transform into an antinucleon, nor decay into exclusively light particles. The concept of nucleonic charge is especially useful for describing the V_1^0-particle which decays into a proton and a pion. The V_1^0-particle, and in general all "heavy nucleons", must have nucleonic charge $+\gamma$.

2. NOTATION

We mention the question of notation, because the literature discussing the single and associated production of particles contains many ambiguous expressions. Any virtual transformation of a nucleon, occurring as a first-order process in perturbation theory, we shall represent by the following general scheme:

A		B		C	
(nucleon)	→	(one particle with nucleonic charge $+\gamma$)	+	(one or more particles with total nucleonic charge zero)	(1)

The order of magnitude of the admixture of the state $(B + C)$ in the state of the nucleon defines the development-parameter of the perturbation theory. In the case when the system C consists of a single particle, the development-parameter may be written in the form $g^2/\hbar c$, where g has the dimension of charge. In this terminology the expressions "single production", "double production", etc., will refer only to the term C of scheme (1).

It should be emphasized that the scheme (1) describes only reactions which have non-vanishing matrix elements in the first approximation. For example, the Yukawa process

$$(N) \rightarrow (N) + (\pi) \quad (N = \text{nucleon})$$

describes "single production" of pions.

3. LIFETIME OF HEAVY MESONS

We shall begin with the question of the neutral pion decay. The neutral pion decay is generally considered as a 3-step process. The first step, the creation of a virtual proton-antiproton pair, follows directly from the possibility of the Yukawa process. In third-order perturbation theory the meson decay is described as follows:

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* This paper is based on the results of work completed in 1951-1953 and already published in reports of the Institute of Nuclear Problems[1,2]. The exposition will follow the second report[2], and the nomenclature for the various particles which was current at that time will be retained.

[1] B. M. Pontecorvo, Report Inst. Nuclear Prob., Acad. Sci., USSR (1951)

[2] B. M. Pontecorvo, Report Inst. Nuclear Prob., Acad. Sci., USSR (1953)

[3] Ia. B. Zel'dovich, Dokl. Akad. Nauk SSSR 86, 505 (1952)

$$\pi^0 \xrightarrow[g_\pi^2]{} (P_1 + \tilde{P}_2) \xrightarrow[e^2]{} (P_2 + \gamma_1 + \tilde{P}_1) \xrightarrow[e^2]{} \gamma_1 + \gamma_2. \quad (2)$$

The probability $(1/t)_{\pi^0}$ of process (2) occurring per unit time must involve the product $(g_\pi^2/\hbar c) \times (e^2/\hbar c)^2$ The interaction of the neutral pion with the electromagnetic field, which is manifested in the two-photon decay, arises from its coupling to the "proton vacuum". In a similar way we shall later use the idea of the coupling of other mesons to the "nucleon vacuum" in order to draw conclusions about the production mechanism of these mesons.

A very interesting fact to which we must give our attention is the following: there exist several types of mesons which are produced with high probability and have a relatively long lifetime ($t \gg 10^{-10}$ sec) against decay into pions. We denote such particles* by the letter τ and call them mesons of the class τ. As we shall now show, it follows from the definition that mesons of the class τ cannot be created singly in virtual processes in which a nucleon remains a nucleon. In other words, the fundamental reaction of the nucleon in which mesons of class τ are created cannot arise from the scheme

$$(N) \rightarrow (N) + (\tau). \quad (3)$$

In fact, if mesons of the class τ (for example, τ^+-mesons) were produced by the process (3), we could write

$$\tau^+ \xrightarrow[g'_\tau]{} (P_1 + \tilde{n}_1) \xrightarrow[g'_\pi]{} (n_1 + \pi_1^+ + \tilde{n}_1) \xrightarrow[g'_\pi]{} \quad (4)$$
$$\rightarrow (n_1 + \pi_1^+ + P_1 + \pi_2^+) \xrightarrow[g'_\pi]{} \pi^+ + \pi^+ + \pi^-.$$

Here the last three steps go with the parameter $(g_\pi^2/\hbar c)$, and the first goes with the parameter $(g_\tau^2/\hbar c)$ which describes the strength of the coupling between the class τ meson and the nucleon. Assuming scheme (3) to hold, the parameter $(g_\tau^2/\hbar c)$ can be roughly estimated from the known ratio of the production probabilities of pions and mesons of class τ. This gives $(g_\tau^2/g_\pi^2) \gtrsim 0.01$. A comparison of processes (2) and (4) then shows that the lifetime of class τ mesons should be shorter than that of the neutral pion (10^{-14} sec), in sharp contradiction with experiment. Therefore, process (3) is not responsible for the production of class τ mesons, unless the production cross section of class τ mesons has such an extreme dependence on energy that the use of the principle of detailed balance is unjustified.

Moreover, the long lifetime of the charged pion itself ($t_{\pi^\pm} \sim 2 \times 10^{-8}$ sec) shows that there does not exist a particle ϵ, with mass less than say $0.3\ m_\pi$, strongly coupled to the nucleon by the scheme

$$(N) \rightarrow (N) + (\varepsilon).$$

4. INADEQUACY OF A PROPOSAL FOR INDIRECT PRODUCTION OF CLASS τ MESONS

At first glance, the contradiction between the abundant production and long lifetime of class τ mesons might seem to be resolved by postulating a heavier meson θ, which would decay with emission of a class τ meson and would be produced according to the scheme

$$(N) \rightarrow (N) + (\theta). \quad (5)$$

However, on this hypothesis, an argument similar to the one which led to the earlier paradox shows that the rate of decay of the θ-meson into pions must be very large. This decay rate must in turn be comparable with the rate of the θ-meson decay which emits a class τ meson, in order that the latter particle be observed. Then, through the intermediate creation of a virtual θ-meson, the class τ meson would decay rapidly into pions, in contradiction with experiment. Hence, the process (5) cannot describe the (indirect) production of class τ mesons.

5. POSSIBLE EXPLANATIONS OF THE LONG LIFETIME OF CLASS τ MESONS

To resolve the contradiction between the abundant production and the long lifetime of class τ mesons, we consider two possibilities for the production process:

a) The class τ meson is produced in pairs with another particle, for example,

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* The following[4] are mesons of the class τ: τ-meson ($m_\tau = 980\ m_e$), V_2^0-meson ($m_{V_2^0} \approx 800\ m_e$), χ-meson ($m_\chi \sim 1500\ m_e$), ζ-meson ($m_\zeta = 540\ m_e$). It is probable that the mesons with mass 500-600 m_e observed by Alikhanian et al[5] also belong to the class τ. But this cannot be stated with certainty, since it is not known whether these mesons decay into pions.

[4] See, for example, R. Daniel et al, Phil. Mag. **43**, 753 (1952)

[5] A. I. Alikhanian, J. Exper. Theoret. Phys. USSR **21**, 1062 (1951)

$$(N)\to(N)+(\tau+?), \qquad (6)$$

however, this possibility, production of integer-spin particles in pairs, seems very artificial.

b) Mesons of class τ are produced singly, but the associated particles with nucleonic charge $+\gamma$ are not ordinary nucleons:

$$(N)\to(\text{heavy nucleon})+(\tau), \qquad (7)$$

In this case, the mesons can clearly be produced abundantly and yet have a long lifetime. This scheme seems logically the most satisfactory. We will return to it after considering the origin of V_1^0-particles.

6. LIFETIME OF V_1^0-PARTICLES

Just as for class τ mesons, there is a contradiction between the abundant production and long lifetime of V_1^0-particles. The probability of the process

$$V_1^0\to P+\pi^- \qquad (8)$$

is only 3×10^9 sec^{-1}. It is obvious that the V_1^0-particle cannot originate in the same reaction which produces pions. In other words, a reaction of the type

$$(N)\to(V_1^0)+(\pi) \qquad (9)$$

is not responsible for the production of V_1^0-particles. This conclusion is made with the same reservation, about the applicability of the principle of detailed balance, as the similar conclusion in Sec. 3 about the production of class τ mesons.

The proposal that the V_1^0-particle is produced together with a pair of pions

$$(N)\to(V_1^0)+(\pi+\pi).$$

is equally unsatisfactory. It would be difficult to imagine a mechanism which would allow production of pions in pairs and forbid single production.

We discuss next the case in which two particles are created with opposite nucleonic charge. There are the following possibilities*

$$\begin{array}{lllll} & A & & B & C \\ \text{a)} & (N) & \to & (N) & +(V_1^0+\tilde{V}_1^0), \\ \text{b)} & (N) & \to & (N) & +(V_1^0+\tilde{N}), \\ \text{c)} & (N) & \to & (V_1^0) & +(V_1^0+\tilde{N}). \end{array}$$

Reaction (b) cannot be responsible for the production of V_1^0-particles. By an argument similar to that used earlier, it can be shown that (b) implies a lifetime of the V_1^0-particle much shorter than is observed. Reaction (a) implies an equally abundant production of V_1^0 and anti-V_1^0-particles. If anti-V_1^0-particles were actually produced, it should not be especially difficult to observe them. The fact that they have not yet been observed speaks against reaction (a).

According to experiment[6], two V_1^0-particles are not commonly produced in a single act. Hence, reaction (c) also cannot be responsible for the production of V_1^0-particles.

Therefore, it is reasonable to postulate that in the production of V_1^0-particles according to scheme (1) there appears a second particle, different from the pion, having integer spin and zero nucleonic charge:

$$\begin{array}{lllll} A & & B & & C \\ (N) & \to & (V_1^0) & + & \text{(Charged meson different from the pion, with nucleonic charge zero).} \end{array} \qquad (10)$$

It is interesting to notice that in Sec. 5 we arrived at a similar scheme (7) in order to explain the long lifetime of the class τ meson.

7. CONNECTION BETWEEN CLASS τ MESONS AND V-PARTICLES

It is natural to identify the "heavy nucleon" in Eq. (7) with the V_1^0-particle and the "charged integer-spin meson" in Eq. (10) with the class τ meson. Then we obtain the following reaction scheme.

$$(N)\to(V_1^0)+(\tau). \qquad (11)$$

This removes simultaneously the contradiction between abundant production and long lifetime, both for particles of class V and class τ. Our argument assumes a moderately large value for the development parameter corresponding to reaction (11). Nucleonic reactions in which pions are produced together with V-particles, and reactions in which class τ mesons are produced without V-particles, are expected to be extremely improbable. Schematically,

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* For simplicity, we do not here envisage the existence of heavy charged V-particles.

[6] R. Leighton et al, Phys. Rev. 89, 148 (1953)

$$(N) \overset{\text{yes}}{\rightarrow} (V_1^0) + (\tau), \qquad (12)$$

$$(N) \overset{\text{no}}{\rightarrow} (V_1^0) + (\pi),$$

$$(N) \overset{\text{no}}{\rightarrow} (N) + (\tau).$$

It must be mentioned that in photographs[6] of the V_1^0-decay, tracks of decaying charged mesons of class τ have not been seen. If scheme (12) is correct, this probably means that the class τ meson is either neutral, or is charged and has a lifetime longer than 10^{-9} sec (the length of track before decay must be longer than the size of the Wilson chamber). In this connection, the observation[5,7] of charged mesons with mass greater than 500 m_e, and with a lifetime 10^{-8} - 10^{-9} sec, is of interest.

The symbol V_1^0 in (11) and (12) denotes the well-known neutral particle which decays into proton and pion. By "heavy nucleons of the class V" we shall mean any long-lived particles which have nucleons among their decay-products. Then we may postulate the more general reaction scheme *

$$\begin{matrix} A & & B & & C \\ (N) & \rightarrow & (V) & + & (\tau) \end{matrix} \qquad (13)$$

It would be premature to consider the precise identification of particles B and C in (13).

The scheme (13) is in some respects similar to the scheme of Yukawa. The class τ meson will cause interaction between nucleons and V-particles, just as the pion causes interaction between ordinary nucleons. This idea is interesting in connection with the possible existence of metastable systems composed of nucleons and V-particles, which will be discussed in the next Section.

8. METASTABLE SYSTEMS COMPOSED OF ORDINARY AND HEAVY NUCLEONS

In the recent literature[9] there have appeared reports of a very interesting phenomenon in high-energy nuclear interactions. In photographic emulsions three cases have been observed of stars which contained, in addition to normal tracks, a single track of a particle which came to rest in the emulsion and gave rise to a secondary star at the point where it stopped. Such a track, in the opinion of the authors, is caused by a "nucleus containing a V_1^0-particle", and the secondary star is caused by the decay of the particle within the nucleus. This proposal deserves attention. As we shall see, the existence of metastable systems composed of nucleons and V-particles is definitely to be expected, if the scheme (13) is correct.

We consider the interaction between a particle of the class V (for simplicity, we shall speak about the V_1^0-particle) and a nucleon, assuming that the rules (12) apply. Since in (12) a class τ meson is responsible for a strong interaction between nucleons and V_1^0-particles, the collision cross section between a V_1^0-particle and a nucleon must be relatively large. When the relative energy of a V_1^0-particle and nucleon is small, only elastic collisions are possible *, while at sufficiently high energies there will also be inelastic collisions accompanied by the creation of a class τ meson. If the forces between nucleon and V_1^0-particle are attractive, then the V_1^0 particle under favorable conditions may be bound with nucleons into a quasi-nuclear system.

We now estimate the degree of stability of such a quasi-nuclear system. The mean energy of a particle bound within the sytem is determined by the temperature of the quasi-nucleus, and is therefore small in comparison with the energy required according to (12) for an inelastic collision of a nucleon and a V_1^0-particle, i.e., a collision in which a class τ meson is produced. Thus, a V_1^0-particle in the quasi-nucleus does not lose its individuality, and decays essentially as if it were free.

We consider now the observed events[9] in more detail. In our scheme a V_1^0-particle is produced together with a class τ meson. The track of the meson is probably one of the relativistic tracks in the shower. The V_1^0-particle is rapidly slowed down by multiple elastic scattering in nuclear matter until further inelastic processes are impossible. This phase of the phenomenon is similar to

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* Similar schemes have been proposed independently by several other authors. See, for example, Pais[8].

[7] J. Astbury et al, Phil. Mag. 44, 242 (1953)

[8] A. Pais, Phys. Rev. 86, 663 (1952)

[9] M. Danysz and J. Pniewski, Phil. Mag. 44, 348 (1953); D. Tidman et al, Phil. Mag. 44, 350 (1953); J. Crussard and D. Morellet, Comptes Rendus 236, 64 (1953)

* The long lifetime of the V_1^0-particle implies that it cannot decay into a nucleon and a class τ meson. If (12) is correct, this decay is forbidden only by energy conservation. In other words, the mass of a class τ meson occurring in (12) must be greater than $(m_{V_1^0} - m_p)$, i.e., $m_\tau > 400\ m_e$.

the well-known process of the nucleonic cascade in nuclear matter. Afterwards, a light quasi-nucleus is emitted containing the V_1^0-particle. At this stage the phenomenon is similar to the emission of nuclear fragments (He, Li, B, etc.) from excited nuclei. Next, the quasi-nucleus is slowed down by ionization energy-loss. The slowing-down time is short compared with the lifetime of the V_1^0-particle, which therefore decays within a stationary quasi-nucleus.

It is worth noting that process of the type $(N) \rightarrow (V_1^0) + (\pi)$ cannot be consistent with the stability of systems composed of nucleons and V_1^0-particles. We already found that the existence of such a process would lead to difficulties in explaining the long lifetime of the free V_1^0-particle. If we admit such a process, it seems completely impossible to imagine a mechanism which would allow the V_1^0-particle to survive for a long time in nuclear matter.

Consequently, if the existence of metastable systems composed of nucleons and V-particles is confirmed, it provides a powerful argument in support of the scheme (13).

CONCLUSIONS

1) Starting from the fact that in high-energy collisions there is abundant production of mesons (of the class τ) which have a long lifetime and decay into pions, one can deduce that the production of these mesons cannot proceed by the scheme

$$(N) \rightarrow (N) + (\tau) \quad (N \equiv \text{nucleon}).$$

2. Similarly, starting from the fact that in high-energy collisions there is abundant production of heavy nucleons of the class V which have a long lifetime and decay into nucleons and pions, one can deduce that the production process for these particles is not

$$(N) \rightarrow (V) + (\pi).$$

3. It is proposed that class τ mesons and heavy nucleons of the class V are produced together by the scheme

$$(N) \rightarrow (V) + (\tau).$$

Then the difficulties arising from the long lifetimes of V-particles and of class τ mesons are simultaneously resolved. In addition, this scheme predicts a strong interaction, mediated by class τ mesons, between nucleons and V-particles.

4. If the scheme $(N) \rightarrow (V) + (\tau)$ is correct, one must expect under favorable conditions the formation of metastable systems composed of nucleons and V-particles.

In conclusion, I wish to thank I. Ia. Pomeranchuk for several valuable discussions.

Translated by F. J. Dyson
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The Problem of the Interpretation of Dirac's Equation for the Electron

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It is shown that Dirac's equation for the electron can be considered a system of equations for two real spinors. Gauge invariance of the equations will correspond to invariance under spinor transformations of the second kind. Consequently, in Dirac theory, it is not the components of ψ which should be considered as basic quantities, but rather the definite tensors which allow one to find the corresponding real spinors to within the spinor transformation.

THE properties of tensors characterized by two real spinors have been studied in a previous paper[1]. This allows one to investigate the Dirac equation from a new point of view. Although the question of which of the relativistically invariant electron equations is to be considered correct remains unanswered[2], nevertheless such an investigation remains interesting in any case. For example, it will make possible the comparison of the Dirac equation with other types, thus allowing for a better understanding of their specific properties and differences. Since a great number of works have been devoted to the Dirac equation, comparison with them can help in the investigation and solution of other equations. On the other hand, the question of the meaning and nature of the Dirac equation may also be posed, which necessitates a thorough investigation of particular situations connected with it*.

1. First, let us note the well-known fact that if two systems of four-by-four matrices are given which satisfy the relations

$$\frac{1}{2}(R^\alpha R^\beta + R^\beta R^\alpha) = g_0^{\alpha\beta}, \qquad (1)$$

$$(g_0^{11} = g_0^{22} = g_0^{33} = -g_0^{44} = 1,\ g_0^{\alpha\beta} = 0 \text{ for } \alpha \neq \beta),$$

$$\frac{1}{2}(\tilde{R}^\alpha \tilde{R}^\beta + \tilde{R}^\beta \tilde{R}^\alpha) = g_0^{\alpha\beta}, \qquad (2)$$

then these matrices are equivalent, i.e.,

$$\tilde{R}^\alpha = O^{-1} R^\alpha O. \qquad (3)$$

(The proof can be found, for example, in references 3 to 5.) The R^α, here, can of course have complex elements.

We emphasize particularly the meaning of transformations of the form of Eq. (3). In previous works[6,7] we treated the quantities R^α as matrices, corresponding to normalized basis vectors in four-dimensional pseudo-Euclidean space. Instead of the R^α, however, we can just as well use the matrices $\tilde{R}^\alpha$, since in the theory of matrix tensors the existence only of relations such as (1) is important. Here we shall deal with a different law of correspondence: to the same basis vectors will correspond matrices of a different form, though the basis vectors themselves remain invariant. We note in particular that one must not confuse a transformation from one coordinate system to another with a transformation from one isomorphic correspondence between matrices and four-vectors to another. These are altogether different things, though in both cases the formulas for expressing the new R^α in terms of the old can have the same form [e.g., Eq. (3)].

2. In Dirac theory, $m_0 c/\hbar$ is considered a scalar. It follows from this that the equation

$$R^\alpha \frac{\partial}{\partial x^\alpha} \psi_{(w)} = \nabla \psi_{(w)} = \frac{m_0 c}{\hbar} J \psi_{(w)}, \qquad (4)$$

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* For example, it necessitates consideration from a new point of view of the generalization of the Dirac equation to the case of the general theory of relativity. The author intends to devote a future paper to the investigation of real spinors in curvilinear coordinates and pseudo-Riemanian spaces, which may lead to the solution of this problem.

[1] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 29, 166 (1955).

[2] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 28, 530 (1955); Soviet Phys. 1, 491 (1955)

[3] H. Weyl, *Gruppentheorie und Quantenmechanik*, 2nd ed., Leipzig, 1931.

[4] B. L. Van der Waerden, *Group Theoretical Methods in Quantum Mechanics*

[5] W. Pauli, *General Principles of Wave Mechanics*

[6] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 25, 667 (1953)

[7] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 28, 524 (1955); Soviet Phys. 1, 411 (1955)

(where the matrices R^{α} are real and $\psi_{(u)}$ is a real spinor) is relativistically invariant.

The Dirac equation for the electron in the presence of an electromagnetic field can be written in the form

$$\left(i\beta\frac{\partial}{\partial x^4} + i\beta\alpha^k\frac{\partial}{\partial x^k}\right)\psi = \frac{m_0c}{\hbar}\psi \quad (x^4 = ct,\ \alpha^k = \alpha_k) \tag{5}$$

(see Pauli[5], Sokolov[8], etc.). We will show that this reduces to two equations of type (4). To do this, let

$$i\beta = \tilde{R} = \tilde{R}^1\tilde{R}^2\tilde{R}^3,\ -i\beta\alpha^k = \tilde{J}\tilde{R}^k; \tag{6}$$

$$(\tilde{J} = \tilde{R}^1\tilde{R}^2\tilde{R}^3\tilde{R}^4,\ k = 1,2,3),$$

so that Eq. (5) becomes

$$\tilde{R}^{\alpha}\frac{\partial\psi}{\partial x^{\alpha}} = \frac{m_0c}{\hbar}\tilde{J}\psi. \tag{7}$$

Transforming to real matrices R^{α} according to Eq. (3), and putting

$$O\psi = \psi_{(1)} + i\psi_{(2)} \tag{8}$$

($\psi_{(1)}$ and $\psi_{(2)}$ are column vectors of real elements), we get

$$R^{\alpha}\frac{\partial}{\partial x^{\alpha}}(\psi_{(1)} + i\psi_{(2)}) = \frac{m_0c}{\hbar}J(\psi_{(1)} + i\psi_{(2)}). \tag{9}$$

The matrices α_k and β, and therefore $\tilde{R}^k$ and $i\tilde{R}^4$, are usually chosen Hermitian (see, for example, Pauli[9]). If R^k and iR^4 are chosen according to Table 1 of reference 6, they will also be Hermitian. Making use of the fact that any matrix which commutes with all the R^{α} is a multiple of the unit matrix, and taking the Hermitian conjugate of both sides of Eq. (3), we get $OO^* = O^*O = kE$. Choosing an appropriate multiplier for O, we get

$$O^* = O^{-1} \tag{10}$$

By way of an example of transition to real quantities, we consider the case of $R^k = IB(0)S_k$, $R^4 = IB(0)$, $J = I$, $R = B(0)$, and use the usual forms,

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is easy then to prove that (12)

$$O = \frac{1}{\sqrt{8}}\begin{pmatrix} 1-i & -1-i & 1-i & 1+i \\ 1+i & 1-i & -1-i & 1-i \\ -1-i & 1-i & -1-i & -1+i \\ -1+i & -1-i & 1-i & -1-i \end{pmatrix},$$

$$O^* = O^{-1}$$

$$= \frac{1}{\sqrt{8}}\begin{pmatrix} 1+i & 1-i & -1+i & -1-i \\ -1+i & 1+i & 1+i & -1+i \\ 1+i & -1+i & -1+i & 1+i \\ 1-i & 1+i & -1-i & -1+i \end{pmatrix},$$

so that $\alpha_k = O^{-1}S_kO$, $i\beta = \tilde{R} = O^{-1}B(0)O$.

We shall show that $\psi_{(1)}$ and $\psi_{(2)}$ can be considered real spinors such as those considered in reference 6. It is first necessary to show that if $\psi_{(1)}$ and $\psi_{(2)}$ transform like real spinors, and ψ like a Dirac-theory spinor, then both sides of Eq. (8) transform in the same way. We shall make use, for the proof, of the notation used by Sokolov[8], so that

$$\beta = \alpha_0 = \rho_3,\quad \tilde{R}^k = \alpha_0\sigma_k, \tag{13}$$

$$\tilde{R}^4 = i\rho_2,\quad \tilde{J} = i\rho_1.$$

We shall also borrow from that reference certain formulas involving ψ. We first consider a spatial rotation. Rotation about the z-axis is characterized in Dirac theory by the matrix $\cos\phi/2 - i\sigma_3\sin\phi/2 = \cos\phi/2 - \tilde{R}\tilde{R}_3\sin\phi/2$ [here ϕ is the angle of rotation about the z-axis in the positive direction; we have used Eq. (18.11) on p. 96 of Sokolov[8], taking into account Eq. (13)]. Real spinors transform in the same way (see the remarks in reference 6 before Eq. (9); only $\tilde{R}^{\alpha}$ must be changed to R^{α}). Analogous considerations hold true for Lorentz transformations [this follows from a comparison of Eqs. (9)-(13) of reference 6 with those on pp. 93-95 of reference 8]. From this it follows that under a four-dimensional rotation, both sides of Eq. (8) transform in the same way. This will be true also for four-dimensional reflections, if the law of reflection of space or time is defined according to one of the last two cases of Pauli[9]. In this way it becomes possible to consider $\psi_{(1)}$ and $\psi_{(2)}$ as real spinors.

Equation (9) can be broken up into two parts by equating the real and imaginary parts. Since $\psi_{(1)}$

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[8] A. A. Sokolov and D. D. Ivanenko, *Quantum Theory of Matter*, GITTL, 1952

[9] W. Pauli, *Relativistic Theory of Elementary Particles*

and $\psi_{(2)}$ do not transform into each other under any four-dimensional rotations or reflections, Eq. (9) determines two equations for two unrelated real spinors $\psi_{(1)}$ and $\psi_{(2)}$, which can be written in the form

$$R^{\alpha}\frac{\partial\psi_{(w)}}{\partial x^{\alpha}}=\frac{m_0c}{\hbar}J\psi_{(w)},\ w=1,2. \tag{14}$$

In this way it is seen that the Dirac equation for empty space is a combination of two independent equations containing separate real spinors, joined into one equation in a purely formal way.

3. Let us go on to a consideration of the general Dirac equation in the presence of an arbitrary external electromagnetic field. As is well-known, this equation (or, more accurately, system of equations) can be written in the form

$$\tilde{R}^{\alpha}\left(\frac{\partial}{\partial x^{\alpha}}+i\frac{e}{\hbar c}A_{\alpha}\right)\psi=\frac{m_0c}{\hbar}\tilde{J}\psi. \tag{15}$$

Here e is the electronic charge and $A = -\phi$.

Taking into account Eqs. (3) and (8), we can rewrite Eq. (15) in the form of a system of equations containing real spinors:

$$R^{\alpha}\frac{\partial\psi_{(1)}}{\partial x^{\alpha}}-\frac{e}{\hbar c}A_{\alpha}R^{\alpha}\psi_{(2)}=\frac{m_0c}{\hbar}J\psi_{(1)}, \tag{16}$$

$$R^{\alpha}\frac{\partial\psi_{(2)}}{\partial x^{\alpha}}+\frac{e}{\hbar c}A_{\alpha}R^{\alpha}\psi_{(1)}=\frac{m_0c}{\hbar}J\psi_{(2)}. \tag{17}$$

This system of equations is clearly exactly equivalent to the Dirac equation.

Equation (14) is a gauge invariant. Thus, Eqs. (16) and (17) do not change their form under a spinor transformation of the second kind ($\psi_{(+)}$ is replaced by $e^{-if}\psi_{(+)}$), so long as A_α is replaced by $A_\alpha + \frac{\hbar c\,\partial f}{e\,\partial x^\alpha}$. In Dirac theory, the only tensor components which are bilinear forms in $\psi_{(1)}$ and $\psi_{(2)}$ that have physical meaning are those which are invariant under a spinor transformation of the second kind. Therefore, according to the preceding article[1], we can explain this situation by noting that the initial primary quantities are Ω_1, Ω_2 and the components $P_{(+)}$, N, and $F_{(+)}$*. Making use of the general theory of tensors, we can derive relations between the quantities under consideration (see reference 1), which, in particular, allow us to express $F_{(+)}$ in terms of $P_{(+)}$, N, Ω_1 and Ω_2. Some of these relations were previously known (see, for example, DeBroglie[10]), but their true meaning remained unclear.

Equations developed in the preceding article[1], which allow one to find the real spinors $\psi_{(1)}$ and $\psi_{(2)}$ in terms of the primary quantities, can be written in the form

$$P_{(+)}\psi_{(+)}=P_{(+)}(\psi_{(1)}+i\psi_{(2)})=2iN\psi_{(+)} \tag{18}$$

and, consequently,

$$F_{(+)}\psi_{(+)}=-2i(\Omega_1-\Omega_2J)\,\psi_{(+)}. \tag{19}$$

We can, of course, go back from $\psi_{(+)}$ to ψ, which corresponds to a transformation to a new isomorphic correspondence between matrices and tensors. Then, for example, instead of Eq. (19), we have $0^{-1}F_{(+)}0\,\psi = -2i\,(\Omega_1 - \Omega_2\tilde{J})\,\psi$, where $0^{-1}F_{(+)}0$ is very simply expressed in terms of the matrices α_k, β and the components of the tensor $F_{(+)}$.

In accordance with the above, the spinors which occur in the Dirac equation must be considered as secondary quantities. Therefore, we arrive at the conclusion that the Dirac equation must be considered a system of equations for definite four-dimensional tensors.

We must note that the notation we are using for writing the Dirac equation only with matrices and real elements is not new. The same notation was used, for example, by Majorana[11] (see also Pauli[9], Kramers[12], Markov[13], etc.). What is new is that the real spinors in this equation are considered secondary quantities, a system of parameters characterizing definite four-dimensional tensors. Accordingly, in the Dirac equation written in any form, the components of ψ must be considered secondary quantities defined by certain four-dimensional tensors.

4. In conclusion, we make some remarks concerning the difference between the Dirac equation and relativistically invariant differential equations of the first degree, containing two real spinors, which were discussed in reference 2.

The basic difference lies in the fact that these equations are invariant under different kinds of spinor transformations. In connection with this, in each case different tensors are considered primary. This difference is closely connected with the fact that, in the earlier work[2], $\hbar$ is considered a pseudoscalar, not merely a scalar, as in Dirac theory, and that the operators of four-"momentum" have entirely different forms.

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* Also, the components of the external electromagnetic field and tensors which are invariant under spinor transformation of the second type and expressed in terms of the A_α, are components of real spinors and their derivatives.

[10] Louis de Broglie, *The Magnetic Electron*, 1936

[11] E. Majorana, Nuvo Cimento **14**, 171 (1937)

[12] H. A. Kramers, Proc. Amst. Acad. Sci. **40**, 814 (1937)

[13] M. A. Markov, J. Exper. Theoret. Phys. USSR **21**, 761 (1951)

The different character of the two systems of differential equations is especially explicit in the transition to the nonrelativistic limit. From the point of view of the earlier work, we deal only with one real spinor, $\psi_{(1)}$. It is very characteristic that the current vector used in nonrelativistic quantum mechanics turns out to be not part of a vector, but of a tensor. Its components are proportional to $T^{(1)}_{4k}$ [see reference 1, Eq. (54) and Zaitsev[14], Eq. (59)].

As for the Dirac equation, in the transition to the equations of nonrelativistic mechanics, the situation is entirely different. In the nonrelativistic limit, ψ is still expressed in terms of two real spinors $\psi_{(1)}$ and $\psi_{(2)}$. The components of the current vector are found from the components of $P_{(+)}$ after making use of the Dirac equation and eliminating some of the terms (see Pauli[5])

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[14] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR **25**, 653 (1953)

Translated by E. J. Saletan

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The Statistics of Charge-Conserving Systems and Its Application to the Theory of Multiple Production

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The quantum statistics of systems with a variable number of non-interacting particles is generalized to the case of an aggregate of oppositely charged particles, which obey the law of charge conservation. Formulas which differ from the corresponding formulas of ordinary quantum statistics are derived for the total number of particles and the total energy. The results obtained are applied to the theory of multiple production of mesons. The following questions are studied: the dependence of the energy on the relative proportions of neutral and charged mesons, the formation of nucleon-antinucleon pairs, and the relation between the yield and the primary energy. The theory is compared with the available experimental data.

1. INTRODUCTION

IN the statistical treatment of the phenomenon of multiple production of particles at high energies, proposed by Fermi[1], the total number of particles, the total energy of the system, and also the relation between the numbers of particles of different sorts in the "thermodynamic" approximation are calculated by the usual quantum statistical formulas for an ideal Bose or Fermi gas with a variable number of particles. However, in this case, it is more appropriate to use formulas which take into account the conservation of charge (electronic, nuclear, etc). This is particularly important when we consider processes with a low yield. Thus, after generalizing ordinary quantum statistics to the case of charge-conserving systems, a more detailed examination of processes of multiple production in the framework of the "thermodynamic" approximation is possible.

We make this generalization in the present paper, and as a result obtain new formulas for the total number of particles and the total energy, which we relate to the corresponding formulas of ordinary statistics. The results obtained are used to explain several matters pertaining to the theory of multiple production of particles.

2. CALCULATION OF THE PARTITION FUNCTION, THE AVERAGE NUMBER OF PARTICLES, AND THE AVERAGE ENERGY OF CHARGE CONSERVING SYSTEMS

We shall consider an ideal gas, consisting of like but oppositely charged particles. The number of positive and negative particles can change as a result of pair production, but the difference between n^+, the number of positive particles, and n^-, the number of negative particles, i.e., the total charge ν of the system, remains constant as a result of charge conservation

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[1] E. Fermi, *Elementary Particles*, New Haven, 1951

$$\nu = n^+ - n^- = \text{const.} \tag{1}$$

If the system has non-degenerate energy levels E_k, then, designating by n_k^+ and n_k^- the corresponding occupation numbers, we can write the partition function of the system in the form[2]

$$Z_1 = \sum_{\substack{n_1^+ \\ n_1^-}} \sum_{\substack{n_2^+ \\ n_2^-}} \dots \prod_k q_k^{n_k^+ + n_k^-} \delta\left[\nu, \sum_k (n_k^+ - n_k^-)\right], \tag{2}$$

where $q_k = e^{-E_k/\theta}$, $\Theta = kT$, and $\delta[a,b]$ is the Kronecker symbol

$$\delta[a, b] = \begin{cases} 1, & \text{if} \quad a = b. \\ 0, & \text{if} \quad a \neq b. \end{cases} \tag{3}$$

The product $\prod\limits_k$ and the summation $\sum\limits_k$ extend over all k from 1 to ∞; the summation over n_k^+ and n_k^- extends from 0 to ∞ in the case of Bose statistics, and from 0 to 1 in the case of Fermi statistics.

As is well-known, the Kronecker symbol can be represented in the form

$$\delta[a, b] = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(a-b)\varphi}\, d\varphi. \tag{4}$$

In this way the partition function can be written as the following integral:

$$Z = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu\varphi} \sum_{\substack{n_1^+ \\ n_1^-}} \sum_{\substack{n_2^+ \\ n_2^-}} \dots \prod_k q_k^{n_k^+ + n_k^-} e^{i(n_k^+ - n_k^-)\varphi}\, d\varphi$$
$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu\varphi} \prod_k \sum_{\substack{n_k^+ \\ n_k^-}} q_k^{n_k^+ + n_k^-} e^{i(n_k^+ - n_k^-)\varphi}\, d\varphi. \tag{5}$$

The sums in the integrand are easily calculated for both Bose and Fermi statistics. Indeed

$$\sum_{\substack{n_k^+ \\ n_k^-}} q_k^{n_k^+ + n_k^-} e^{i(n_k^+ - n_k^-)\varphi} = \sum_{n_k^+} (q_k e^{i\varphi})^{n_k^+} \sum_{n_k^-} (q_k e^{-i\varphi})^{n_k^-}$$
$$= (1 \pm q_k e^{i\varphi})^{\pm 1} (1 \pm q_k e^{-i\varphi})^{\pm 1} \tag{6}$$

Here, as in what follows, the upper sign (+ in the example given) applies to the case of Fermi statistics, and the lower sign (– in the example given) to the case of Bose statistics.

Using Eq. (6), we can write the partition function (5) in the form

$$Z = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu\varphi + \Phi(\varphi)}\, d\varphi, \tag{7}$$

where

$$\Phi(\varphi) = \pm \sum_k \ln(1 \pm q_k e^{i\varphi}) \pm \sum_k \ln(1 \pm q_k e^{-i\varphi}). \tag{7'}$$

Having determined Z, we can calculate the average occupation numbers by the well-known formula

$$\bar{n}_k = \overline{n_k^+} + \overline{n_k^-} = q_k \frac{\partial}{\partial q_k} (\ln Z). \tag{8}$$

Using Eqs. (7), (7′), and (8), we obtain

$$\bar{n}_k = \frac{1}{2\pi Z} \int_0^{2\pi} \left(\frac{1}{1 \pm \exp\left\{\frac{E_k}{\Theta} + i\varphi\right\}} + \frac{1}{1 \pm \exp\left\{\frac{E_k}{\Theta} - i\varphi\right\}} \right) e^{-i\nu\varphi + \Phi(\varphi)}\, d\varphi. \tag{9}$$

Knowing Z and $\bar{n}_k$, we can calculate all the more important thermodynamic quantities.

To calculate Z and $\bar{n}_k$, we must find an approximate expression for $\Phi(\varphi)$. If in the energy interval dE there are, on the average, $dG(E)$ energy levels, $\Phi(\varphi)$ can be approximated as

$$\Phi(\varphi) = \pm \int_0^\infty \left[\ln\left(1 \pm \exp\left\{-\frac{E}{\Theta} + i\varphi\right\}\right) + \ln\left(1 \pm \exp\left\{-\frac{E}{\Theta} - i\varphi\right\}\right)\right] dG(E). \tag{10}$$

Since the difference between the statistics we are considering and statistics with a fixed number of particles can be appreciable only for energies large enough to produce pairs, we restrict ourselves to considering a relativistic gas. For an ideal relativistic gas,

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[2] M. A. Leontovich, *Statistical Physics*, GTTI, Moscow, 1944

$$dG(E)=\frac{b\Omega}{2\pi^2c^3\hbar^3}E^2\,dE, \tag{11}$$

where Ω is the volume in which the gas is contained, and b the number of possible states of different polarization. Substituting Eq. (11) in Eq. (10) and integrating, we obtain

$$\Phi(\varphi)=\frac{b\Omega}{2\pi^2c^3\hbar^3}\,4\,\Theta^3\sum_{k=1}^{\infty}(\mp 1)^{k-1}\frac{\cos k\varphi}{k^4}. \tag{12}$$

In view of the very rapid convergence of the series obtained, we take only its first term, and obtain as an approximation for $\Phi(\varphi)$ the expression

$$\Phi(\varphi)=\rho\cos\varphi,\ \text{where}\ \rho=2\,b\,\Omega\Theta^3/\pi^2c^3\hbar^3. \tag{13}$$

This approximation corresponds to the transition to Boltzmann statistics.

Substituting Eq. (13) in Eq. (7), we obtain finally

$$Z=\frac{1}{2\pi}\int_0^{2\pi}e^{-i\nu\varphi+\rho\cos\varphi}d\varphi=I_\nu(\rho), \tag{14}$$

where $I_\nu(\rho)$ is the Bessel function of order ν with imaginary argument.

The average number of particles N and the average energy W of the system can be calculated by the formula

$$N=\int_0^{\infty}\bar{n}(E)\,dG(E),\ W=\int_0^{\infty}E\bar{n}(E)\,dG(E), \tag{15}$$

where $n(E)$ is the average number of particles occupying levels with energy E, i.e., $\bar{n}_k$ in the notation of Eq. (9).

To calculate N, we substitute Eq. (9) in Eq. (15) use the expression (12), and integrate first over E, whence we obtain

$$N=\frac{\rho}{2\pi Z}\int_0^{2\pi}\sum_{k=1}^{\infty}(\mp 1)^{k-1}\frac{\cos k\varphi}{k^3} \tag{16}$$

$$\times\exp\left\{-i\nu\varphi+\rho\sum_{k=1}^{\infty}(\mp 1)^{k-1}\frac{\cos k\varphi}{k^4}\right\}d\varphi.$$

Just as in the calculation of Eq. (14), we restrict ourselves to the first terms of the series in the integrand, and obtain

$$N=\frac{\rho}{I_\nu(\rho)}\frac{dI_\nu(\rho)}{d\rho}. \tag{17}$$

By analogous calculations, we also obtain an expression for the average energy

$$W=3\Theta N. \tag{18}$$

For conciseness, the formulas obtained above for an ideal gas, consisting of positive and negative particles, will be hereafter referred to as the formulas of charge statistics.

3. AN ANALYSIS OF THE FORMULAS OF CHARGE STATISTICS

The dimensionless parameter ρ appearing in Eq. (17) has a simple physical interpretation. According to Eq. (13), ρ is (except for a multiplier near unity) the average number of particles of relativistic gas with $2b$ internal degrees of freedom, calculated by the rules of the usual quantum statistics of systems with a variable number of particles. Therefore ρ is a parameter suitable for comparing formulas (17) and (18) of charge statistics with the corresponding formulas of the usual quantum statistics.

From the theory of Bessel functions, we know that

$$\rho\frac{dI_\nu(\rho)}{d\rho}-\nu I_\nu(\rho)=\rho I_{\nu+1}(\rho), \tag{19}$$

whence, by Eq. (17)

$$N=\nu+\rho\frac{I_{\nu+1}(\rho)}{I_\nu(\rho)}. \tag{20}$$

Thus N is the sum of two terms: ν, the minimum number of particles for the given system, equal to the number of excess charges of whatever sign, and the mean number of produced particles

$$N_{\pm}=\rho\frac{I_{\nu+1}(\rho)}{I_\nu(\rho)}=\rho\lambda_\nu(\rho). \tag{21}$$

The quantity $\lambda_\nu(\varphi)$ denotes the ratio of the number of produced particles, calculated by charge statistics, to the number of particles produced according to the usual statistics. From the asymptotic behavior of the Bessel functions, it follows that $\frac{I_{\nu+1}(\rho)}{I_\nu(\rho)}\to 1$ as $\rho\to\infty$. Thus $\lambda_\nu(\rho)\to 1$ for sufficiently high temperatures (but bounded values of ν), i.e., charge statistics reduces asymptotically to ordinary quantum statistics. On the other hand, for $\rho\to 0$, i.e., for low temperatures, $\lambda_\nu(\rho)\to 0$, and the formulas of charge statistics are considerably different from the formulas of ordinary statistics.

The quantity $\lambda_\nu(\rho)$ can be approximated with good accuracy (especially in regions of small ρ) by the expression

$$\lambda_\nu(\rho)=I_{\nu+1}(\rho)/I_\nu(\rho) \tag{22}$$

$$= 1 - \exp\{-\rho/2(\nu+1)\}.$$

From this formula it is clear that the number of particles produced at a given temperature (i.e., for given ρ) drops sharply as the charge ν is increased, in other words, the charge ν acts as a kind of anticatalyst in the process of particle production. For sufficiently large values of ν, practically no particles are produced, the average number of particles remains constant, and charge statistics reduces to the ordinary statistics of systems with a constant number of particles.

Of particular interest is the case $\nu = 0$. According to (22), for temperatures that are not too high $(\rho < 1)$

$$\lambda_0(\rho) = 1 - e^{-\rho/2} \cong \rho/2. \qquad (23)$$

Thus, for relativistic but not too high temperatures, the average number of charged particles produced is equal to $\rho^2/2$, while the average number of neutral particles produced is equal to $\rho/2$. This leads to different temperature dependences for the average energy of radiation for a gas of neutral particles ($W \sim \Theta^4$) and for a gas of particles with some kind of charge ($W \sim \Theta^7$).

4. APPLICATIONS TO THE FERMI THEORY OF MULTIPLE PRODUCTION OF MESONS AND NUCLEON-ANTINUCLEON PAIRS

According to Fermi's hypothesis[1], π-mesons and nucleon-antinucleon pairs are produced as a result of nucleon-nucleon collisions in some small volume Ω. The number of these particles is calculated by formulas valid for systems in thermodynamic equilibrium, using the expressions of ordinary quantum statistics, and not those of charge statistics, which, as we have seen, give quite different results for the case of low yield. For very large energies, the charge ν is small compared to the number of particles produced, $\lambda_\nu(\rho) \to 1$, and the formulas of charge statistics coincide with the formulas of ordinary statistics; thus, in this region of energies, the application of charge statistics in the framework of the Fermi hypothesis can not lead to new results. Significantly different results are obtained when charge statistics are applied in the low energy region, where the average number of particles produced is comparable to the charge ν of the system.

In Fermi's calculation of the average number of charged π-mesons, the same formulas are used as in the calculation of the number of neutral mesons. However, the charged π-mesons, unlike the neutral π-mesons, obey charge statistics, and therefore to calculate their average number it is necessary to use the formulas (21) and (22), and not the formula $N = \rho$, valid only for π^0-mesons. According to Eq. (22), we obtain for the ratio of the number of neutral π-mesons to the number of charged π-mesons, produced as a result of nucleon-nucleon collisions

$$\alpha = \frac{1}{2(1 - e^{-\rho/2})}, \qquad (24)$$

since the number of neutral π-mesons produced is equal to $\rho/2$. This ratio coincides with that obtained by Fermi only for $\rho \to \infty$. For small values of ρ, it is clear that α increases, which agrees with the experimental evidence[3,4]*

Because of the conservation of nuclear charge[5], it is also necessary to use the formulas of charge statistics to calculate the number of nucleons and antinucleons produced by nucleon-nucleon collisions. Thus, it follows from Eqs. (21) and (22) that the number of nucleons and antinucleons produced (within the framework of the Fermi hypothesis) by the collision of two nucleons, is given by the formula

$$N_2 = \rho(1 - e^{-\rho/6}), \qquad (25)$$

and not by the formula $N_2 = \rho$, used by Fermi. According to Eq. (25), the number of nucleon-antinucleon pairs produced must be considerably smaller than that given by the Fermi formulas. This number becomes even smaller if the Fermi hypothesis is applied to the collisions of nucleons with nuclei, and it is assumed that the energy is distributed among several nucleons of the nucleus. If we neglect the production of particles in subsequent nucleon-nucleon and meson-nucleon collisions, the number of nucleons and antinucleons produced must be calculated by the formula

$$N_{1+A} = \rho\,[1 - \exp\{-\rho/2(A+2)\}], \qquad (26)$$

where A is the number of colliding nucleons. In this case, the ratio of the number of nucleon-antinucleon pairs to the number of mesons can be expressed by the formula

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* In the case where the average number of particles produced $N \cong 1$, $\rho/2$ is small, and we may consider that $W = (\rho/2) + (\rho^2/2)$, $\alpha = 1/\rho$. Eliminating ρ, we find that $\alpha \cong 1$.

[3] A. G. Carlson, I. E. Hooper, and D. T. King, Phil. Mag. 41, 701 (1950)

[4] U. Camerini, P. H. Fowler, W. O. Lock, and H. Muirhead, Phil Mag. 41, 413 (1950)

[5] E. P. Wigner, Proc. Nat. Acad. Sci. 38, 449 (1952)

$$\beta = {}^{8}/_{3}\,[1 - \exp\{-\rho/2(A+2)\}], \tag{27}$$

where ρ is taken for a gas of nucleons and antinucleons ($b = 4$, since the polarization states of both the neutrons and protons are considered). For high energies and small A , this ratio reduces to Fermi's, namely, $\beta = 8/3$, as was to be expected; however, for not too high energies and large A, it is considerably smaller. To calculate β in terms of the given primary energy W, it suffices to know ρ as a function of W. For this, we must use Eq. (18), generalized to the case of the system considered, in analogy to the way we treat the case of nucleon-nucleon collisions below.

We shall derive the dependence on the primary energy $N_{\pm}$, the average number of charged mesons produced in the process of nucleon-nucleon collisions without charge exchange. We designate by γ the total energy of the primary nucleon in the center of mass system, expressed in units of the nucleon rest mass, by $\gamma_0 = 2\gamma^2 - 1$ the same energy in the laboratory system, and by γ_k the energy, incident on the nucleon in the center of mass system and corresponding to the threshold of meson production. The medium energy region is pertinent to charge statistics, where $\gamma_0 < 10$ and, correspondingly, $\gamma < 2.4$. In this case, as experiment shows, the probability of producing nucleon-antinucleon pairs is very small, and we shall neglect it. The production of secondary particles will also be neglected.

Using Eqs. (17) and (18), where $\nu = 0$,

$$W = 2(\gamma - \gamma_k),\ \Theta = 0.1\gamma^{1/3}\rho^{1/3} \quad \text{[see (13]}$$

we obtain

$$\rho^{4/3}\left[\lambda_0(\rho) + \frac{1}{2}\right] = 6.7\,\frac{\gamma - \gamma_k}{\gamma^{1/3}}, \tag{28}$$

$$N_{\pm} = \rho\lambda_0(\rho). \tag{29}$$

Eliminating ρ from Eqs. (28) and (29) (using graphs) we get the dependence of the number of charge mesons produced on the primary energy. In the energy region considered, this dependence can be written

$$N_{\pm} = k(\gamma - \gamma_k), \tag{30}$$

where $k \cong 2.67$. If we put $\gamma_k = 1.1$, Eq. (30) gives satisfactory agreement with experiment[6].

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[6] J. G. Wilson editor, *Progress in Cosmic Ray Physics* Vol. 1, New York, 1952

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The Quantum Theory of Fields I

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A simple derivation is carried out of the equations proposed by Schwinger[1]. The resultant equations are discussed and renormalization is obtained in the original integral equations*.

INTRODUCTION

In connection with the inapplicability of the usual excitation theory for large energies of interacting particles, and also for the calculation of the characteristic properties of the existing theories of quantum fields (the presence of infinities, the asymptotic character of a series of excitation theories, the asymptotic character of the Green's function for high momenta, etc.), the development of more precise methods of solution of the quantum mechanical equations presents great interest. In this connection, equations of the form suggested by Schwinger possess fundamental interest. Inasmuch as we want to solve these equations not only by the methods of ordinary excitation theory, it becomes essential to carry out the renormalization (not according to the excitation theory) in the original equations. Moreover, even within the framework of ordinary excitation theory, it is of interest to obtain a system of equations which does not preserve the infinities. On the other hand, to clarify the physical meaning of quantities which enter into the equations, obtained by Schwinger from a variational principle, and to establish the boundary conditions which must be imposed on the solutions of these equations, it is important to obtain these equations from the ordinary scheme of the theory of quantum fields, without resorting to a variational principle. Such a derivation of the desired equations is given in the present work, and the renormalization is carried out in the integral equations that are obtained.

1. We consider a quantum mechanical system which is a generalization of the ordinary system. Let the Lagrangian L of the system consist of the ordinary Lagrangian L^{or} and the Lagrangian of the interaction with external sources L^{in}.

For definiteness, we consider the case of quantum electrodynamics (the results are simply carried over to the general case of quantum mechanical theory). The initial Lagrangian in this case has the form:

$$L = L^{\mathrm{or}} + L^{\mathrm{in}}; \tag{1}$$

$$L^{\mathrm{or}} = -{}^1/_4\,[\bar{\psi}, \gamma_\mu(-i\partial_\mu - eA_\mu)\psi + m\psi] + {}^1/_4 F_{\mu\nu}^2 - {}^1/_4\{F_{\mu\nu}, \partial_\mu A_\nu - \partial_\nu A_\mu\} + \text{Hermitian conjugate}; \tag{1a}$$

$$L^{\mathrm{in}} = {}^1{}_2/\,[\bar{\psi}, \eta] + {}^1/_2\,[\bar{\eta}, \psi] + J_\mu A_\mu, \tag{1b}$$

where J is the external source of the photon field; η is the anti-commutating external source of the spinor field;

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\delta_{\mu\nu}; \tag{1c}$$

$$\delta_{11} = \delta_{22} = \delta_{33} = 1; \ \delta_{44} = -1.$$

The Hamiltonian of the system can be written in the form

$$H = H^{\mathrm{or}} + H^{\mathrm{in}}, \tag{2}$$

where H^{in} has the form

$$H^{\mathrm{in}} = -\int \{{}^1/_2\,[\bar{\psi}\eta] + {}^1/_2\,[\bar{\eta}\psi] + J_\mu A_\mu\}\, d^3x. \tag{3}$$

Conditionally we write H^{in} in the form

$$H^{\mathrm{in}} = -\sum_n \int \pi_{f_n}(x, t)\, f_n(x, t)\, d^3x, \tag{4}$$

where f_n is the source of an nth type field (J_μ, η, $\bar{\eta}$).

In the Heisenberg representation, where the wave function of the system is independent of time, $\Psi(t) = \mathrm{const}$, the equations of motion for the operators of the field have the form

$$\gamma_\mu(-i\partial_\mu - eA_\mu)\psi + m\psi = \eta; \tag{5}$$

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* The renormalization was obtained by us in reference 2; in the present paper, a more appropriate derivation of the renormalized system of equations is given.

[1] J. Schwinger, Proc. Nat. Acad. Sci. U. S. 37, 452 (1951)

[2] E. S. Fradkin, J. Exper. Theoret. Phys. USSR 26, 751 (1954)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \tag{6}$$

$$-\Box A_\mu + \partial_\mu \partial_\nu A_\nu = J_\mu + j_\mu;$$

$$(\partial A_\mu / \partial x_\mu) = \psi = 0, \tag{7}$$

where

$$j_\mu = {}^1/_2\, e[\bar{\psi}, \gamma_\mu \psi]. \tag{8}$$

We transform to a new representation, where the operators of the field do not depend on the external sources (analogous to the representation of the interaction in external interaction) and have the form

$$\gamma_\mu(-i\partial_\mu - e\tilde{A}_\mu)\tilde{\psi} + m\tilde{\psi} = 0; \tag{9}$$

$$-\Box \tilde{A}_\mu + \partial_\mu \partial_\nu \tilde{A}_\nu = \tilde{j}_\mu. \tag{10}$$

The wave function in the new representation changes with time and can be written in the form:

$$\Phi = S(t, -\infty)\Psi_0, \tag{11}$$

where Ψ_0 is the value of the wave function for $t = -\infty$, and S satisfies the equation

$$i\frac{\partial S(t, -\infty)}{\partial t} = \tilde{H}^{\text{in}} S(t, -\infty). \tag{12}$$

We construct the matrix $S(t_1, t_2)$

$$S(t, t') = S(t, -\infty) S^{-1}(t', -\infty). \tag{13}$$

In accordance with Eqs. (12) and (13), we get the equation for $S(t_1, t_2)$ in integral form:

$$S(t, t') = 1 - i\int_{t'}^{t} H^{\text{in}}(t_1) S(t_1, t')\, dt_1 \tag{14}$$

$$= 1 + i\int_{t'}^{t}\int \bar{\pi}_{f_n}(t_1, x_1) f_n(x_1, t_1) S(t_1, t')\, d^4x_1.$$

We take the functional derivative of both parts of Eq. (14) with respect to any external source $f_n(\xi)$ at the point ξ, first setting $t' = -\infty$*,

$$\frac{\delta S(t, -\infty)}{\delta f(\xi)} = \begin{cases} i\tilde{\pi}_{f_n}(\xi) S(\xi_0, -\infty) + i\int_{t_0}^{t} \tilde{H}^{\text{in}}(t_1) \dfrac{\delta S(t, -\infty)}{\delta f_n(\xi)}\, dt_1, & \text{when } t > \xi_0, \\ 0, & \text{when } t < \xi_0. \end{cases} \tag{15}$$

Multiplying Eq. (14) by $i\pi_{f_n}(\xi_0) S(\xi_0, -\infty)$ on the left, and setting $t' = \xi$, we get

$$iS(t, \xi_0)\tilde{\pi}_{f_n}(\xi_0) S(\xi_0, -\infty) \tag{16}$$

$$= i\tilde{\pi}_{f_n}(\xi) S(\xi_0, -\infty)$$

$$+ i\int_{\xi_0}^{t} \tilde{H}^{\text{in}}(t_1) S(t_1, \xi)\, i\pi_{f_n}(\xi_0) S(\xi, -\infty)\, dt_1.$$

Comparing Eqs. (15) and (16), we obtain

$$\frac{\delta S(t, -\infty)}{\delta f(\xi)} \tag{17}$$

$$= \begin{cases} iS(t, \xi_0)\tilde{\pi}_{f_n}(\xi) S(\xi_0, -\infty), & t > \xi_0; \\ 0, & t < \xi_0. \end{cases}$$

But, as is well-known, an arbitrary operator $\tilde{\pi}$ is connected with the corresponding operator in the Heisenberg representation by the relation

$$S^{-1}(t, -\infty)\tilde{\pi} S(t, -\infty) = \pi(t). \tag{18}$$

We get, finally,

$$\frac{\delta S(t, -\infty)}{\delta_{f_n}(\xi)} \tag{19}$$

$$= \begin{cases} iS(t, -\infty)\pi_{f_n}(\xi), & t > \xi_0; \\ 0, & t < \xi_0. \end{cases}$$

It is easy to show that the following general formula exists for functional differentiation with respect to the external source $f_n(\xi)$:

$$\frac{\delta S(\infty)\tilde{F}(x)}{\delta_{f_n}(\xi)} \tag{20}$$

$$= \frac{\delta}{\delta f_n(\xi)}[S(\infty) S^{-1}(t) F(x) S(t)]$$

$$= iS(\infty) P'_{x_0\xi_0}[F(x)\pi_{f_n}(\xi)],$$

where

$$P'_{x_0\xi_0}[F(x)\pi_{f_n}(\xi)] \tag{21}$$

$$= \begin{cases} F(x)\pi_{f_n}(\xi), & \text{if} \quad x_0 > \xi_0; \\ \pm\pi_{f_n}(\xi) F(x), & \text{if} \quad x_0 < \xi_0. \end{cases}$$

The plus sign is used in Eq. (21) when the source f commutes with the operator $F(x)$, and the minus sign in the opposite case [e.g., $f_n = \eta$, $F(x) = \psi$].

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* This method was applied in reference 3 for the case of a photon field and Eq. (15) was obtained, although the S matrix in reference 3 is different from ours (it considers interactions not only with external sources).

[3] R. Utiyma et al, Progr. in Theor. Phys. 81 (1953); K. Yamaraki, Progr. in Theor. Phys. 7, 449 (1952)

In particular, we get from Eqs. (19) and (20) the formulas

$$\frac{\delta S(\infty)}{\delta\eta(x)} = -iS(\infty)\,\bar{\psi}(x); \qquad (22)$$

$$\frac{\delta S(\infty)}{\delta\eta(x)\,\delta J(x')} = +S(\infty)\,P(\psi(x)\,\dot{A}(x'));$$

$$\frac{\delta S(\infty)}{\delta\bar{\eta}(x)} = iS(\infty)\,\psi(x);$$

$$\frac{\delta S(\infty)}{\delta\bar{\eta}(x)\,\delta J(x')} = -S(\infty)P(\psi(x)\,A(x')); \qquad (23)$$

$$\frac{\delta S(\infty)}{\delta J_\mu(x)} = iS(\infty)\,A_\mu(x);$$

$$\frac{\delta S(\infty)}{\delta\bar{\eta}(x)\,\delta\eta(x')} = S(\infty)\,P'(\psi(x)\,\bar{\psi}(x'));$$

$$\prod_{l=1}^{n}\frac{\delta}{\delta J(x_n)}\,\frac{\delta}{\delta\bar{\eta}(x)}\,S(\infty) \qquad (24)$$

$$= (i)^n\,S(\infty)\,P\left[\prod_{i=1}^{n}\dot{A}(x_n)\,\psi(x)\right] \text{ etc.}$$

With the aid of the formulas obtained for the function derivatives of $S(\infty)$, we can write down the operator equations (5) to (8) in compact form and get the same functional equation for the determination of the operators $S(\infty)$. For this purpose we multiply Eqs. (5) - (8) on the left by $iS(\infty)$: from Eq. (23), we get

$$\left[\gamma_\mu\left(-i\partial_\mu + ie\frac{\delta}{\delta J_\mu(x)}\right) - m\right]\frac{\delta S(\infty)}{\delta\bar{\eta}(x)} \qquad (25)$$

$$= i\eta(x)\,S(\infty);$$

$$\left(-\Box\frac{\delta S(\infty)}{\delta J_\mu} + \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\frac{\delta S(\infty)}{\delta J_\nu}\right) \qquad (26)$$

$$= +iJ_\mu(x)\,S(\infty)$$

$$+\frac{ie}{2}\,\mathrm{Tr}\,\gamma_\mu\left[\frac{\delta^2}{\delta\bar{\eta}(x)\,\delta\eta(x+\varepsilon)}\right.$$

$$\left.+\frac{\delta^2}{\delta\bar{\eta}(x)\,\delta\eta(x-\varepsilon)}\right]S(\infty); \quad \varepsilon\to 0;$$

$$\frac{\partial}{\partial x_\mu}\,\frac{\delta S(\infty)}{\delta J_\mu(x)}\,\Psi = 0. \qquad (27)$$

The system of operator equations (25) - (27) must be solved for the following boundary conditions:

$$\left[\frac{\delta S(\infty)}{\delta\bar{\eta}_\alpha(x,t)},\ \frac{\delta S(\infty)}{\delta\eta_\beta(x',t)}\right]_{+\,J=0,\ \eta=0} \qquad (28)$$

$$= -\gamma_0\delta_{\alpha\beta}\delta(x-x');$$

$$\left[\frac{\delta S(\infty)}{\delta\eta(x,t)},\ \frac{\delta S(\infty)}{\delta\eta(x',t)}\right]_{+\,J=0,\ \eta=0}$$

$$= \left[\frac{\delta S(\infty)}{\delta\bar{\eta}(x,t)},\ \frac{\delta S}{\delta\bar{\eta}(x',t)}\right]_+ = 0;$$

$$\left[\frac{\delta S(\infty)}{\delta J_\mu(x,t)},\ \frac{\partial}{\partial t}\,\frac{\delta S(\infty)}{\delta J_\nu(x',t)}\right]_{J=0,\ \eta=0} \qquad (29)$$

$$= \delta_{\mu\nu}\delta(x-x');$$

$$\left[\frac{\delta S(\infty)}{\delta J_\mu(x,t)},\ \frac{\delta S}{\delta\eta(x',t)}\right]_-$$

$$= \left[\frac{\delta S(\infty)}{\delta J_\mu(x,t)},\ \frac{\delta S}{\delta\bar{\eta}(x',t)}\right]_- = 0;$$

$$\Psi^* S^*(\infty)\,S(\infty)\,\Psi = 1.$$

From Eqs. (22) - (24) it is easy to prove that if the matrix element of transition from $S(\infty)$ is found in the vacuum state for $t = -\infty$ and for $y = +\infty$, then we find the matrix elements of the transitions in the successive functional differentiations with respect to the external sources. Therefore, instead of solving the operator equations (25) - (28), it suffices to solve the system of equations obtained from these equations for the matrix element of transition vacuum-vacuum*:

$$(\Phi_0^*(+\infty)\,\Phi_0(-\infty)) \qquad (30)$$

$$= (\Psi_0^*(-\infty)\,S(\infty)\,\Psi_0(-\infty)) = Z.$$

From Eqs. (25) - (27) and (30), we obtain

$$\left[\gamma_\mu\left(-i\partial_\mu + ie\frac{\delta}{\delta J_\mu}\right) + m\right]\frac{\delta Z}{\delta\bar{\eta}(x)} = i\eta(x)\,Z; \qquad (31)$$

$$\left(+i\partial_\mu + ie\frac{\delta}{\delta J_\mu}\right)\frac{\delta Z}{\delta\eta}\,\gamma_\mu \qquad (31a)$$

$$+ m\,\frac{\delta Z}{\delta\eta} = -i\bar{\eta}(x)\,Z;$$

$$-\Box\frac{\delta Z}{\delta J_\mu} + \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\frac{\delta Z}{\delta J_\nu} = iJ_\mu(x)\,Z \qquad (32)$$

$$+\lim_{\varepsilon\to 0}\frac{ie}{2}\,\mathrm{Tr}\,\gamma_\mu\left[\frac{\delta^2 Z}{\delta\eta(x)\,\delta\eta(x-\varepsilon)}\right.$$

$$\left.+\frac{\delta^2 Z}{\delta\eta(x-\varepsilon)\,\delta\eta(x)}\right];$$

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* By functional differentiation with respect to external sources [see Eqs. (22) - (24)], and by then setting $\eta = J = 0$, we can obtain all the matrix elements for the effects, with consideration of all radiative corrections, from Eq. (30).

$$\frac{\partial}{\partial x_\mu}\frac{\delta Z}{\delta J_\mu}=0. \qquad (33)$$

From Eqs. (30) - (33) it is easy to obtain the system of equations suggested by Schwinger[1]. For this purpose, we consider that the matrix element of an arbitrary operator $F(x)$ (in the notation of Schwinger) is

$$\langle F(x)\rangle=\frac{\Psi_0^* S(\infty) F(x) \Psi_0}{\Psi_0^* S(\infty)\Psi_0}, \qquad (34)$$

where Ψ_0 describes the vacuum state. In view of this equation, Eqs. (31) - (34) take on the form

$$\gamma_\mu\left(-i\partial_\mu - e\langle A_\mu\rangle + ie\frac{\delta}{\delta J_\mu}\right)\langle\frac{\delta Z}{\delta\eta(x)}\rangle \qquad (35)$$

$$+ m\langle\frac{\delta Z}{\delta\eta(x)}\rangle = i\eta(x);$$

$$\left(i\partial_\mu - e\langle A_\nu\rangle + ie\frac{\delta}{\delta J_\mu}\right)\langle\frac{\delta Z}{\delta\eta(x)}\rangle\gamma_\mu \qquad (36)$$

$$+ m\langle\frac{\delta Z}{\delta\eta(x)}\rangle = -i\bar{\eta}(x);$$

$$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu} - \Box\delta_{\mu\nu}\right)\langle\frac{\partial Z}{\partial J_\mu}\rangle \qquad (37)$$

$$= iJ_\mu(x) + \lim_{\varepsilon\to 0}\frac{ie}{2}\,\mathrm{Tr}\,\gamma_\mu$$

$$\langle\left[\frac{\delta^2}{\delta\eta(x-\varepsilon)\,\delta\eta(x)} + \frac{\delta^2}{\delta\eta(x)\,\delta\eta(x-\varepsilon)}\right]Z\rangle;$$

$$\frac{\partial}{\partial x_\mu}\langle\frac{\delta Z}{\delta J_\mu}\rangle = 0 \qquad (38)$$

with boundary conditions

$$\langle\frac{\delta Z}{\delta\eta(x)}\rangle = \langle\frac{\delta Z}{\delta\bar{\eta}(x)}\rangle = \langle\frac{\delta Z}{\delta J_\mu}\rangle = 0, \qquad (38a)$$

when

$$J=0; \qquad \eta=0; \qquad \bar{\eta}=0$$

and with the condition that

$$\left.\frac{\delta^2 Z}{\delta\eta(x)\,\delta\eta(x')}\right|_{\eta=0\ J=0}; \qquad \left.\frac{\delta^2 Z}{\delta J_\mu(x)\,\delta J_\nu(x')}\right|_{J=0,\,\eta=0}$$

take on only positive frequencies $x_0 > x_0'$ and only negative frequencies for $x_0 < x_0'$.

Following Schwinger, we introduce the following definitions:

$$\left.-i\frac{\delta}{\delta\eta(x')}\langle\frac{\delta Z}{\delta\eta(x)}\rangle\right|_{\eta=0} = G(x,x')$$

$$= i\ \langle P'(\psi(x),\bar{\psi}(x'))\rangle;$$

$$-i\frac{\delta}{\delta J_\mu(x')}\langle\frac{\delta Z}{\delta J_\nu(x)}\rangle = D_{\mu\nu}(x,x')$$

$$= i\,[P\langle A_\mu(x)A_\nu(x')\rangle - \langle A_\mu(x)\rangle\langle A_\nu(x')\rangle].$$

G and D are the Green's functions of the electron and photon fields, respectively, From Eqs. (35) - (38) we obtain the following system of equations:

$$\left[\gamma_\mu(-i\partial_\mu - e\langle A_\mu\rangle) + ie\gamma_\mu\frac{\delta}{\delta J_\mu}\right] \qquad (39)$$

$$\times\ G(x,x') + mG(x,x') = \delta(x-x');$$

$$\left[i\partial'_\mu - e\langle A_\mu(x')\rangle + ie\frac{\delta}{\delta J_\mu(x')}\right] \qquad (40)$$

$$\times\ G(x,x')\gamma_\mu - mG(x,x')$$

$$= -\delta(x-x');$$

$$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu} - \Box\delta_{\mu\nu}\right)\langle A_\mu\rangle \qquad (41)$$

$$= J_\mu + \lim_{x'\to x}\mathrm{Tr}\,\gamma_\mu\left[\frac{ie}{2}\left(G(x,x') + G(x',x)\right)\right];$$

$$\left(-\Box\delta_{\mu\nu} + \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\lambda}\right)D_{\lambda\nu}(x,x_1) \qquad (42)$$

$$= \delta_{\mu\nu}\,\delta(x-x_1)$$

$$+ \lim_{x'\to x}\frac{\delta}{\delta J_\nu(x_1)}\mathrm{Tr}\,\gamma_\mu\left[\frac{ei}{2}G(x,x') + G(x',x)\right];$$

$$\frac{\partial}{\partial x_\nu}D_{\mu\nu} = 0.$$

Here $G(x,x')$ and $D(x,x')$ contain only positive frequencies for $x_0 > x_0'$ and only negative frequencies for $x_0 < x_0'$; at the point where $x_0 = x_0'$, these functions are defined as the half sum of the quantities for $x_0' = x_0 \pm \epsilon$.

We transform to the momentum representation, obtaining

$$B(x_1, x_2, \ldots, x_n) \qquad (43)$$

$$= \frac{1}{(2\pi)^{4(n-1)}}\int \exp\ \{ip_1x_1 - ip_2x_2 - \ldots$$

$$- ip_nx_n\}\,B(p_1, p_2, \ldots, p_n)\,d^4p_1\ldots d^np_n$$

$$(B = A,\ G,\ J,\ D).$$

From Eqs. (39) - (43), we obtain the following system of equations

$$(\hat{p} + m)\,G(p,p_1) \qquad (44)$$

$$- e\langle\int\hat{A}(p-k)\rangle\,G(k,p_1)\,d^4k$$

$$+\frac{ie}{(2\pi)^4}\int\gamma_\mu \frac{\delta G(p+k,\ p_1)}{\delta J_\mu(k)}d^4k=\delta(p-p_1); \tag{45}$$

$$k^2\langle A_\mu(k)\rangle = J_\mu(k)$$

$$+\frac{ie}{(2\pi)^4}\,\mathrm{Tr}\left\{\int\gamma_\mu G(p+k,\ p)\,d^4p\right\};$$

$$(k^2\delta_{\mu\lambda}-k_\mu k_\lambda)\,D_{\lambda\nu}(k,\ k_1) \tag{46}$$

$$=\delta_{\mu\nu}\delta(k-k_1)+\frac{ie}{(2\pi)^4}\,\mathrm{Tr}\left\{\int\gamma_\mu\frac{\delta G(p+k,\ p)}{\delta J_\nu(k_1)}\right\}.$$

In what follows, it is advantageous to go over to another form of the equations obtained for the Green's function:

$$(\hat{p}+m)\,G(p,\ p_1) \tag{47}$$

$$-e\int\langle\hat{A}(p-k)\rangle G(k,\ p_1)\,d^4k$$

$$+\int\sum(p,\ k)\,G(k,\ p_1)\,d^4k=\delta(p-p_1);$$

$$(k^2\delta_{\mu\nu}-k_\mu k_\nu)\,A_\nu(k) \tag{48}$$

$$=J_\mu(k)+\frac{ie}{(2\pi)^4}\,\mathrm{Tr}\left\{\int\gamma_\mu G(p+k,\ p)\,d^4p\right\};$$

$$(k^2\delta_{\mu\lambda}-k_\mu k_\lambda)\,D_{\lambda\nu}(k,\ k_1)=\delta_{\mu\nu}\delta(k-k_1)-\int P_{\mu\sigma}(k,\ p)\,D_{\sigma\nu}(p,\ k_1)\,d^4p; \tag{49}$$

where

$$\sum(p,\ k_1)=-\frac{e^2}{(2\pi)^4 i}\int\gamma_\mu\frac{\delta G(p+k,\ k_1)}{\delta J_\mu(k)}G^{-1}(p_1,\ k_1)\,d^4p_1d^4k=$$

$$=-\frac{e^2}{(2\pi)^4 i}\int\gamma_\mu G(p+k,\ p_1)\,\Gamma_\nu(p_1,\ k_1,\ k_2)\,D_{\nu\mu}(k_2,\ k)\,d^4k_2d^4kd^4p_1; \tag{50}$$

$$P_{\mu\nu}(p,\ k_1)=\frac{e^2}{(2\pi)^4 i}\,\mathrm{Tr}\left\{\gamma_\mu\int\frac{\delta G(p+k,\ k)}{\delta e\langle A_\nu(k_1)\rangle}d^4k\right\}=$$

$$=\frac{e^2}{(2\pi)^4 i}\,\mathrm{Tr}\left\{\int\gamma_\mu G(p+k,\ k_2)\,\Gamma_\nu(k_2,\ k_3,\ k_1)\,G(k_3,\ k)\,d^4kd^4k_2d^4k_3\right\}; \tag{51}$$

$$\Gamma_\mu(p,\ k,\ k_1)=-\frac{\delta G^{-1}(p,\ k)}{\delta e\langle A_\mu(k_1)\rangle}=\gamma_\mu\delta(p-k-k_1)-\frac{\delta\Sigma(p,\ k)}{\delta e\langle A_\mu(k_1)\rangle}\,. \tag{52}$$

From this set of equations we can, in particular, obtain an infinite set of coupled equations by means of successive differentiation of Eqs. (47) - (52) with respect to $A_\mu(k)$.

In the frequent case when the external sources are absent ($J=0$; $\eta=0$), this sytem of coupled equations has the form⁴:

$$\{\hat{p}+m+\varepsilon^{(0)}(p)\}\,G^0(p)=1; \tag{53}$$

$$\{(k^2\delta_{\mu\rho}-k_\mu k_\rho)+P^0_{\mu\rho}(k)\}\,D^0_{\rho\nu}(k)=1; \tag{54}$$

$$\varepsilon^{(n)}_{\mu_1,\ \ldots,\ \mu_n}\left(p,\ p-\sum_{m=1}^{n}s_m,\ s_1,\ \ldots,\ s_n\right) \tag{55}$$

$$\times\ -\frac{e^2}{(2\pi)^4 i}\int\gamma_\mu\sum_{m,\ k=0}^{n}{}^{*}\ G^{(m)}$$

$$\times\left(p+k,\ p+k-\sum_{r=1}^{m}s_{i_r},\ \ldots,\ s_{i_m}\right)$$

$$\times\ \Gamma^{(k)}_{\nu\mu_{i_{m+1}}\ldots\mu_{i_{m+k}}}\left(p+k-\sum_{r=1}^{m}s_{i_r},\ p-\sum_{r=1}^{n}s_r,\right.$$

$$\left.k+\sum_{r=m+k+1}^{n}s_{i_r},\ s_{i_{m+1}},\ \ldots,\ s_{i_{m+k}}\right)\times D^{(n-m-k)}$$

$$\times\left(k+\sum_{r=m+k+1}^{n}s_{i_r},\ k,\ s_{i_{m+k+1}},\ \ldots,\ s_{i_n}\right)d^4k;$$

$$\Gamma^{(n)}_{\mu,\ \mu_1\ldots\mu_n}\left(p+k,\ p-\sum_{m=1}^{n}s_m,\ k,\ s_1\ldots s_n\right) \tag{56}$$

$$=\gamma_\mu\delta_{n0}-\varepsilon^{(n+1)}_{\mu,\ \mu_1\ldots\mu_n}$$

$$\times\left(p+k,\ p-\sum_{m=1}^{k}s_m,\ k,\ s,\ \ldots,\ s_m\right);$$

$$G^{(n)}_{\mu_1\ldots,\mu_n}\left(p,\ p-\sum_{r=1}^{n}s_r,\ s_1\ldots s_n\right) \tag{57}$$

$$=\sum_{m=0}^{n}{}^{*}\,G^{(m)}_{\mu_{i_1}\ldots\mu_{i_m}}\left(p,\ p-\sum s_{i_r},\ s_{i_1}\ldots s_{i_m}\right)$$

$$\times\Gamma^{(n-m-1)}_{\mu_{i_{m+1}}\mu_{i_n}}\left(p-\sum_{r=1}^{m}s_{i_r},\right.$$

$$\left.p-\sum_{r=1}^{n}s_r;\ s_{i_{m+1}}\ldots s_{i_n}\right)G^{(0)}\left(p-\sum_{r=1}^{n}s_r\right);$$

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[4] B. L. Ioffe, Compt. Rend., Acad. Sci., USSR 95, 761 (1954)

$$D^{(n)}_{\rho\nu\mu,\ \mu_n}\left(p,\ p-\sum_{r=1}^{n} s_r,\ s_1\ldots s_n\right) \tag{58}$$

$$=-\sum{}^{*} D^{(m)}_{\mu_{i_1}\ldots\mu_{i_m}}\left(p,\ p-\sum_{n=1}^{m} s_{i_r},\ s_{i_1}\ldots s_{i_m}\right)$$

$$\times P^{(n-m)}_{\rho\nu\mu_{i_{m+1}}\ldots\mu_{i_n}}\left(p-\sum_{r=1}^{m} s_{i_r},\ p-\sum_{r=1}^{n} s_r,\ s_{i_{m+1}}\ldots s_{i_n}\right) D^{(0)}\left(p-\sum_{r=1}^{n} s_r\right);$$

$$P^{(n)}_{\nu\rho\mu_1\ldots\mu_n}\left(p,\ p-\sum s_{i_n},\ s_1\ldots s_{i_n}\right)= \tag{59}$$

$$=\frac{e^2}{i\,(2\pi)^4}\,\mathrm{Tr}\left\{\int\gamma_\nu \sum{}^{*} G^{(m)}_{\mu_{i_1}\ldots\mu_{i_m}}\left(p+k,\ p+k-\sum_{r=1}^{m} s_{i_r},\ s_{i_1}\ldots s_{i_m}\right)\right.$$

$$\times \Gamma^{(k)}_{\rho\mu_{i_m}\ldots\mu_{i_{m+k}}}\left(p+k-\sum_{r=1}^{n} s_{i_r},\ k+\sum_{r=1}^{n} s_{i_r},\ p-\sum_{r=1}^{n} s_r,\ s_{i_{m+1}}\cdot s_{i_{m+k}}\right)$$

$$\times G_{\mu,\ m+k+1}\left(k+\sum_{r=m+k+1}^{n} s_{i_r},\ k,\ s_{i_{m+k+1}}\ldots s_{i_n}\right) d^4k,$$

where $\sum\limits_{r=1}^{m}{}^{*} f(s_{i_r})$ denotes that all the variables i_r take on values from $r=1$ to $r=m$, but such that

$$i_1 \neq i_r \neq \ldots \neq i_m.$$

In drawing up such a program of renormalization, we transform to new variables G', $D'_{\mu\nu}$, Γ'_μ, A'_μ, J'_μ, e', ψ', η', $\bar{\psi}'$, $\bar{\eta}'$ which are related to the former variables in the following way:

$$G'=\frac{G}{Z_2};\qquad D'_{\mu\nu}=\frac{D_{\mu\nu}}{Z_3}; \tag{60}$$

$$\Gamma'_\mu = Z_1\Gamma_\mu;$$

$$\langle A'_\mu\rangle=\frac{\langle A_\mu\rangle}{Z_3^{1/2}};\ \psi'=\frac{\psi}{Z_2^{1/2}},$$

$$\bar{\psi}'=\frac{\psi}{Z_2^{1/2}};\ \eta'=Z_2^{1/2}\eta;\ \bar{\eta}'=Z_2^{1/2}\bar{\eta};\ J'_\mu=Z_3^{1/2}J_\nu.$$

The renormalization reduces to a choice of definite values for the constants Z_1, Z_2, Z_3 and to the substitution for m and e the experimentally observed values for these quantities. This leads to the impositon of the following conditions on the solution of the renormalized equations in the absence of external sources ($J=0$, $\eta=0$).

1) The Green's function for the electron must have a first order pole at the momentum $p=-m_{\exp}$, where $m_{\exp}$ is the experimental value of the electronic mass. The constant Z_2 must be so chosen that it would follow from the equation for G' that

$$G'(p)\rightarrow\frac{1}{\hat{p}+m_{\exp}},\quad \text{when}\quad \hat{p}\rightarrow -m_{\exp}. \tag{61}$$

2) The Green's function of the photon has a pole for momentum $k^2=0$. The constant Z_3 is so chosen that it would follow from the equation for G' that

$$D'_{\mu\mu}(k)\rightarrow 1/k^2 \qquad \text{for} \qquad (k^2)=0. \tag{62}$$

3) It follows from the relativistic invariance of the theory that for $\hat{p}=\hat{p}_0=-m_{\exp}$ the quantity $\Gamma_\mu(p^0, p^0, 0)$ is proportional to γ_μ.

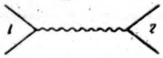
We choose the constant Z_1 so that it would follow from the equation for Γ that

$$\Gamma'(p^0,\ p^0,\ 0)=\gamma_\mu. \tag{63}$$

For these conditions it is not difficult to prove that $e'=Z_1^{-1}Z_2Z_3^{1/2}e$ is equal to the experimental charge $e_{\exp}$. Actually, let us consider any experiment with whose help the value of the electric charge is determined (for example, the scattering of electrons of small momenta). This process is described, to a first approximation, by the excitation theory diagram cited. The matrix element in this approximation can be written in the form

$$e^2\left(\psi_1^{*(0)}\psi_2^{*(0)}\gamma_\mu D^{(0)}_{\mu\nu}\gamma_\nu\psi_1^{(0)}\psi_2^{(0)}\right). \tag{64}$$

Calculation of the radiation corrections reduces to the replacement of the zeroth approximation



ψ^0, γ_μ, $D^{(0)}_{\mu\nu}$ by the values of these quantities with account taken of the radiation corrections to them, i.e., ψ', Γ_μ and $D_{\mu\nu}$, respectively. However, as follows from Eqs. (60) - (63), for such momenta of the electrons, when they are almost free ($-\hat{p} \approx m_{\exp}$) and quanta of low energy are exchanged (the momentum of the photons $k \approx 0$), consideration of the radiation corrections does not change the type of matrix element of the zeroth approximation, but only leads to a replacing of e by $e' = Z_1^{-1} Z_2 Z_3^{1/2} e$ in Eq. (64) in the exact matrix element e'. Comparing the predictions of the theory with experiment, we determine the experimental value of the charge which appears in front of the brackets of the form (64) in the exact matrix element, and this constant is e', as we have shown *.

From Eqs. (47) - (52) and (60), we obtain the following system of equations:

$$Z_2(\hat{p}+m)G'(p,p_1) - e_{\exp} Z_1 \int \langle \hat{A}(p-k)\rangle G'(k,p_1)\, d^4k + \int \sum{}'(p,k)\, G'(k,p_1)\, d^4k = \delta(p-p_1); \tag{65}$$

$$Z_3\{k^2\delta_{\mu\nu} - k_\mu k_\nu\}\langle A'_\mu(k)\rangle = J'_\mu(k) + \frac{ie_{\exp}}{(2\pi)^4} Z_1 \operatorname{Tr}\left\{\int \gamma_\mu G'(p+k,p)\, d^4p\right\}; \tag{66}$$

$$Z_3\{k^2\delta'_{\mu\lambda} - k_\mu k_\lambda\} D'_{\lambda\nu}(k,k_2) + \int P'_{\mu\lambda}(k,p)\, D'_{\lambda\nu}(p,k_1)\, d^4p = \delta(k-k')\,\delta_{\mu\nu}; \tag{67}$$

$$P'_{\mu\nu}(p,k_1) = \frac{e^2_{\exp}}{(2\pi)^4 i} Z_1 \operatorname{Tr}\left\{\int \gamma_\nu G'(p+k,k_2)\,\Gamma'_\nu(k_2,k_3,k_1) \times G'(k_3,k)\, d^4k_2\, dk_3\, dk;\right. \tag{68}$$

$$\sum{}'(p,k_1) = -\frac{e^2_{\exp}}{(2\pi)^4 i} Z_1 \int \gamma_\mu G'(p+k,p_1)\,\Gamma'_\nu(p_1,k_1,k_2)\, D'_{\nu\mu}(k_2 k)\, d^4k_2\, dp_1\, dk; \tag{69}$$

$$\Gamma'_\mu(p,k,k_1) = Z_2\gamma_\mu\delta(p-k-k_1) - \frac{\delta\Sigma'(p,k)}{\delta e'_{\exp}\langle A'_\mu(k_1)\rangle}. \tag{70}$$

The constants Z_1, Z_2, Z_3 are determined from the conditions (61) - (63). To find them, it is appropriate to write down the set (65) - (70) for the case $\eta = 0$, $J = 0$:

$$\left[Z_2(\hat{p}+m) + \sum{}'(p)\right] G'(p) = 1; \tag{71}$$

$$[Z_3(k^2\delta_{\mu\rho} - k_\mu k_\rho) + P'_{\mu\rho}(k)]\, D'_{\rho\nu}(k) = \delta_{\mu\nu}; \tag{72}$$

$$\sum{}'(p) = \frac{-e^2_{\exp}}{(2\pi)^4 i} \times Z_1 \int \gamma_\mu G'(p+k)\,\Gamma'_\nu(p+k,p,k)\, D'_{\nu\mu}(k)\, d^4k; \tag{73}$$

$$P'_{\mu\nu}(p) = \frac{e^2}{(2\pi)^4 i} Z_1 \times \operatorname{Tr}\left\{\int \gamma_\mu G'(p+k)\,\Gamma_\nu(p+k,k,p)\, G(k)\, d^4k\right\}; \tag{74}$$

$$\Gamma'_\mu(p+k,p,k) = Z_1\gamma_\mu - \sum{}^{(1)}_\mu(p+k,p,k) \text{ and so on,} \tag{75}$$

where
$$\sum{}^{(1)}_\mu = \frac{\delta\Sigma'}{\delta e_{\exp}\langle A'_\mu\rangle} \quad \text{for} \quad J = 0. \tag{76}$$

From Eq. (71) we have

$$G'(p) = \frac{1}{Z_2(\hat{p}+m) + \Sigma'(p)} \tag{77}$$

When $-\hat{p} \to m_{\exp}$, expanding $\Sigma'(p)$ in a series in $\hat{p} + m_{\exp}$ we get .

$$G'(p)|_{-\hat{p}\to m_{\exp}} \to \frac{1}{Z_2(\hat{p}+m) + \Sigma'(p^0) + \dfrac{\partial\Sigma'(p^0)}{\partial p^0}(\hat{p}+m_{\exp})}. \tag{78}$$

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* It is a reflection of this fact that in the renormalized equations (see below) for the almost free electrons ($-\hat{p} \approx m_{\exp}$) in the interaction with low energy quanta (momentum $k \approx 0$), there is no radiation correction. As a coupling constant in these equations, we have not e but e', to which, consequently, it is necessary to add the value of the experimental charge.

On the other hand, in accord with Eq. (61) this same quantity must equal $1/(\hat{p}+m_{\mathrm{exp}})$, whence we obtain

$$m = m_{\mathrm{exp}} - \frac{\Sigma'(p^0)}{Z_2}; \quad Z_2 = 1 - \frac{\partial \Sigma'(p^0)}{\partial \hat{p}^0}. \tag{79}$$

Analogously, it follows from Eqs. (62) and (72) that*

$$Z_3 = 1 - \frac{1}{2}\frac{\partial^2 P_{\mu\mu}(k^0)}{\partial k_\nu^0 \partial k_\nu^0}\bigg|_{\hat{k}^0=0} \tag{80}$$

$$= 1 - \frac{\partial P_{\mu\mu}(k^0)}{\partial (k^0)^2}\bigg|_{(k^0)^2=0}$$

Here we have considered the circumstance that 1) $P_{\mu\nu}(k^0)$ is equal to zero (the proper mass of the photon) from the gradient invariance of the theory; 2). from the law of conservation of the Dirac current and the gradient invariance, $P_{\mu\nu}(k) = B(k^2) \times (k^2\delta_{\mu\nu} - k_\mu k_\nu)$, where $B(k^2)$ is some function of the square of the momentum and therefore the first term of the expansion in powers of k is equal to zero for $k = 0$.

Finally, from Eqs. (75) and (63), we get

$$Z_1\gamma_\mu = \gamma_\mu + \Sigma_\mu^{(1)}(p^0, p^0, 0); \tag{81a}$$

$$Z_3 = 1 + \frac{\gamma_\mu}{4}\Sigma_\mu^{(1)}(p^0, p^0, 0). \tag{81b}$$

Substituting Eqs. (79) - (81) in the infinite set of coupled equations (53) - (59), we get the following set of renormalized coupled equations (we omit the primes):

$$[\hat{p} + m_{\mathrm{exp}} + \varepsilon_1^{(0)}(p)]\,G(p) = 1; \tag{82}$$

$$[k^2\delta_{\mu\rho} - k_\mu k_\rho + P_{1\mu\rho}^{0}{}^{(k)}]\,D_{\rho\mu} = 1; \tag{83}$$

$$\varepsilon_1^{(0)}(p) = \varepsilon^{(0)}(p) - \varepsilon^{(0)}(\hat{p}^0) \tag{84}$$

$$- \frac{\partial \varepsilon^{(0)}(p^0)}{\partial \hat{p}^0}(\hat{p} - m_{\mathrm{exp}})\big|_{-\hat{p}^0 = m_{\mathrm{exp}}};$$

$$P_{1\mu\rho}^{(0)}(k) = P_{\mu\nu}(k) - P_{\mu\nu}(0) - k_\lambda \frac{\partial P_{\mu\nu}(k^0)}{\partial k_\lambda^0} \tag{85}$$

$$- \frac{1}{2}k_\lambda k_\rho \frac{\partial P_{\mu\nu}}{\partial k_\lambda^0 \partial k_\rho^0}\bigg|_{(k^0)^2=0};$$

$$\Gamma_\mu^{(0)}(p+k, p, k) = \gamma_\mu \tag{86}$$

$$- \varepsilon^{(1)}(p+k, p, k) + \varepsilon^{(1)}(p^0, p^0, 0);$$

$$\varepsilon^{(n)}_{\mu_1\ldots\mu_n}\Big(p, p - \sum_{m=1}^{n} s_{l_n}, s_1 \ldots s_n\Big) \tag{87}$$

$$= -\frac{e^2_{\mathrm{exp}}}{(2\pi)^4 i} Z_1 \int \gamma_\mu \sum_{m,\,h=0}^{n}{}^{*} G^{*}\Big(p+k, p+k$$

$$- \sum_{r=1}^{m} s_{l_r}, s_{l_r} \ldots s_{l_m}\Big) \times \Gamma^{(h)}_{\nu\mu_{l_{m+1}}}$$

$$\cdots{}_{\mu_{l_{m+h}}}\Big(p + k - \sum_{r=1}^{m} s_{l_r}, p - \sum_{r=0}^{n} s_{l_r},$$

$$k + \sum_{r=m+h+1}^{n} s_{l_r}, s_{l_{m+1}} \ldots s_{l_{m+h}}\Big) \times D^{(n-m-h)}_{\mu\nu}$$

$$\times \Big(k + \sum_{r=m+n+1}^{n} s_{l_r}, k, s_{l_{m+k+1}} \ldots s_{l_n}\Big)\, d^4k;$$

$$\Gamma^{(n)}_{\mu,\mu_1\ldots\mu_n}\Big(p+k, p - \sum_{m=1}^{n} s_m, k, s_1 \ldots s_n\Big) \tag{88}$$

$$= \gamma_\mu \delta_{n_0} - \varepsilon^{(n+1)}_{\mu,\,\mu_1\ldots\mu_n}\Big(p+k, p$$

$$- \sum_{m=1}^{n} s_m, k, s_1 \ldots s_n\Big) + \varepsilon^{(1)}(p^0, p^0, 0);$$

$$G^{(n)}_{\mu_1\ldots\nu_n}\Big(p, p - \sum_{r=1}^{n} s_r, s_1 \ldots s_n\Big) \tag{89}$$

$$= \sum_{m=0}^{n}{}^{*} G^{(m)}_{\mu_{l_1}\ldots\mu_{l_m}}\Big(p, p - \sum_{r=1}^{n} s_{l_r}, s_{l_1} \ldots s_{l_m}\Big)$$

$$\times \Gamma^{(n-m+1)}_{\mu_{l_{m+1}}\ldots\mu_{l_n}}\Big(p - \sum_{r=1}^{m} s_{l_r},$$

$$p - \sum_{r=1}^{n} s_r, s_{l_{m+1}} \ldots s_{i_n}\Big)\, G^{(1)}\Big(p - \sum_{r=1}^{n} s_r\Big);$$

$$D^{(n)}_{\rho\nu\mu_1\ldots\mu_n}\Big(p, p - \sum_{r=1}^{n} s_r, s_1 \ldots s_n\Big) \tag{90}$$

$$= -\sum{}^{*} D_{\nu_{l_1}\ldots\mu_{l_m}}\Big(p, p - \sum_{r=1}^{m} s_{l_r}, s_{l_1} \ldots s_{l_m}\Big)$$

$$\times P^{(n-m-1)}_{\rho\nu\mu_{l_{m+1}}\ldots\mu_{l_n}}\Big(p - \sum_{r=1}^{m} s_{l_r},$$

$$p - \sum_{r=1}^{n} s_r, s_{l_{m+1}} \ldots s_{l_n}\Big)\, D^{(0)}\Big(p - \sum_{r=1}^{n} s_r\Big);$$

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* In Eq. (78) the summation is not carried out over the repeated indices μ.

$$P^{(n)}_{\rho\nu\mu_1\ldots\mu_n}\Big(p,\ p-\sum_{m=1}^{n} s_{l_n},\ s_1\ldots s_n\Big) \tag{91}$$

$$= -\frac{e^2_{\mathrm{exp}}}{(2\pi)^4 i} Z_1\,\mathrm{Sp}\Big\{\int \gamma_\rho \sum^{n\,*}_{m,\,k=0} G\Big(p+k,$$

$$p+k-\sum_{r=1}^{m} s_{l_r},\ s_{l_1}\ldots s_{l_n}\Big)$$

$$\times\,\Gamma^{(k)}_{\nu_{l_{m+1}}\ldots\mu_{l_{m+k}}}\Big(p+k-\sum_{r=1}^{m} s_{l_r},$$

$$k+\sum_{r=m+k+1}^{n} s_{l_r},\ p-\sum_{r=1}^{n} s_r,\ s_{l_{n+1}}\ldots s_{l_n}\Big)$$

$$\times\, G^{(n-m-k)}_{\mu_{l_{m+k+1}}\ldots\mu_{l_n}}\Big(k$$

$$+\sum_{r=m+k+1}^{n} s_{l_r},\ k,\ s_{l_{m+k+1}}\ldots s_{l_n}\Big)\,d^4k\Big\}.$$

From Eqs. (81) - (91), we obtain the following equation for Z_1*

$$Z_1^{-1} = 1 + \frac{e^2_{\mathrm{exp}}}{4(2\pi)^4 i} \tag{92}$$

$$\times \int \gamma_\mu\gamma_\rho\,[G^{(0)}(\hat{\mathrm{p}}+\hat{\mathrm{k}})\,\Gamma_\rho(\hat{\mathrm{p}}^0+\hat{\mathrm{k}};\ \hat{\mathrm{p}}^0+\hat{\mathrm{k}},\ 0)$$

$$\times\, G^{(0)}(\hat{\mathrm{p}}_0+\hat{\mathrm{k}}^0)\,\Gamma^0_\nu(\hat{\mathrm{p}}^0+\hat{\mathrm{k}},\ p^0,\ k)\,D^{(0)}_{\mu\nu}(k)$$

$$+\,G^0(\hat{\mathrm{p}}^0+\hat{\mathrm{k}})$$

$$\times\,\Gamma^{(1)}_{\mu\nu}(\hat{\mathrm{p}}^0+\hat{\mathrm{k}},\ p^0,\ k,\ 0)\,D^{(0)}_{\mu\nu}(k)]\,d^4k.$$

Substituting the quantities Z_1, Z_2, Z_3 in Eqs. (65) - (70), we obtain the set of renormalized equations produced in reference 2.

For the effective exclusion of overlapping infinities of Z_1, it is necessary to express Γ_μ, $\partial G^{-1}/\partial p_\mu$ and $\partial D^{-1}/\partial k_\mu$ by the corresponding (uncited) graph (the role of the cited graphs is that they reduce the γ_μ which enter in the uncited graph to Γ_μ); the very automaticity also effectively leads to the exclusion of the overlapping infinities. In this case we introduce the following quantities*:

$$R^{(J)}_{\mu\lambda}(p,\ s\,|\,p_1,\ s_1) = \frac{e^2}{(2\pi)^4 i}\int \frac{\delta}{\delta e_{\mathrm{exp}}\,\langle A_\mu(s)\rangle} \tag{93}$$

$$\times\,[G(p,\ k)\,\Gamma_\nu(k,\ p_1,\ k_1)\,D_{\nu\lambda}(k_1,\ s_1)]^{(J)}\,d^4k\,d^4k_1 - \frac{e}{(2\pi)^4 i}\int \frac{\delta}{\delta e_{\mathrm{exp}}\,A_\rho(-k_3)}$$

$$\times\,[G(p,\ k)\,\Gamma_\nu(k,\ k_1,\ k_2)\,D_{\nu\lambda}(k_2,\ s_1)]^{(J)}\,R^{(J)}_{\mu\rho}(k_1,\ k_3\,|\,p_1,s)\,d^4k\,d^4k_1 d^4k_2\,d^4k_3;$$

$$K_{\mu\lambda}(p,\ p-k,\ k) = -\frac{e^2}{(2\pi)^4 i}\frac{\partial}{\partial p_\mu}[G(p)\,\Gamma_\nu(p,\ k)\,D_{\nu\lambda}(k)] \tag{94}$$

$$+\frac{e^2}{(2\pi)^4 i}\int [G(p)\,\Gamma_\rho(p,\ -k_1)\,G(p+k_1)\,\Gamma_\nu(p+k_1,\ k) + G(p)\,\Gamma^{(1)}_{\nu\rho}(p,\ k,-k_1)]$$

$$\times\,D_{\nu\lambda}(k)\,K_{\mu\rho}(p+k_1-k,\ k_1,\ p-k);$$

$$Y_{\lambda\mu}(p+k,\ k) = \frac{\partial}{\partial k_\mu}[G(p+k)\,\Gamma_\lambda(p+k,\ k)] \tag{95}$$

$$-\int [G(p+k)\,\Gamma_\rho(p+k,\ -k_1)\,G(p+k+k_1)\,\Gamma_\nu(p+k+k_1,\ k)$$

$$+\,G(p+k)\,\Gamma^{(1)}_{\rho\nu}(p+k,\ k,\ -k_1)]\,D_{\nu\lambda}(k)\,Y_{\lambda\mu}(k_1+p,\ k_1)\,d^4k_1.$$

The set of renormalized equations then takes the following form:

$$\frac{\partial G^{-1}(p)}{\partial p_\mu} = \gamma_\mu + \int \{\Gamma_\lambda(p,\ -k)\,K_{\lambda\mu}(p+k,\ k,\ p) \tag{96}$$

$$-\,\Gamma_\lambda(p_0,\ -k)\,K_{\lambda\mu}(p_0+k,\ k,\ p_0)\}\,d^4k;$$

$$\frac{\partial D^{-1}(k)}{\partial k_\mu} = 2k_\mu \tag{97}$$

$$+\,\frac{e_{\mathrm{exp}}}{(2\pi)^4 i}\Big\{\Pi_\mu(k) - \Pi_\mu(k_0) - k_{0\sigma}\frac{\partial \Pi_\mu(k_0)}{\partial k_{0\sigma}}\Big\};$$

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* Formally, in the brackets in Eq. (92), we must still add one term $G^0(\hat{\mathrm{p}}^0+\hat{\mathrm{k}}) \times \Gamma^0_\nu(p^0+k,\ p_0,\ k) \times D^{(1)}_{\nu\mu\rho}(k,\ k,\ 0)$; however, as was shown in the Appendix, this term is equal to zero.

* Here and below the symbol J denotes that the given quantity is taken for the presence of external photon sources; if this symbol is absent, this means that $J=0$.

$$\Gamma_\mu^{(J)}(p_1, p_2, p_3) = \gamma_\mu \delta(p_1 - p_2 - p_3) \tag{98}$$

$$- \int \{\Gamma_\lambda^{(J)}(p_1 s, -s_1) R^{(J)}(s, s_1 | p_2, p_3)$$

$$- \Gamma_\lambda(p_0, s, -s_2) R_{\mu\lambda}(s, s_1 | p_2, 0)\} \, d^4s \, d^4s_1;$$

$$\Pi_\mu(k) = \frac{1}{3} \operatorname{Tr} \Big\{ \int \Gamma_\lambda(p, -k) \tag{99}$$

$$\times Y_{\lambda\mu}(p + k, k) \, d^4p \Big\};$$

where

$$D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) D(k^2) + \frac{k_\mu k_\nu}{k^2} d_l(k^2), \tag{100}$$

$$k_0^2 = 0, \quad p_0^2 = m^2_{\exp}.$$

Here $G^{-1}(\mathrm{p})$ and $D^{-1}(k)$ satisfy the following boundary conditions:

$$(p_0 - m) G(p_0) = 1, \quad k_0^2 D(k_0^2) = 1. \tag{101}$$

Choosing the set of equations (73) - (76) along uncited graphs (of the same order in charge) for Γ_μ, H_μ, $\partial G^{-1}(\mathrm{p}) / \partial p_\mu$ we obtain a complete set of renormalized equations.

APPENDIX

a) By way of an example of reduction with functional derivatives, we consider the solution of the set of Eqs. (25), (26) in the absence of interaction. Averaging these equations over all states and transforming to the momentum representation, we get

$$(\hat{\mathrm{p}} + m) \frac{\delta \langle S(\infty) \rangle}{\delta \eta(p)} = i\eta(p) \langle S(\infty) \rangle; \tag{A1}$$

$$\frac{\delta \langle S(\infty) \rangle}{\delta \eta(p)} (\hat{\mathrm{p}} + m) = - i\bar{\eta}(p) \langle S(\infty) \rangle;$$

$$\Box \frac{\delta \langle S(\infty) \rangle}{\delta J_\mu} = iJ_\mu \langle S(\infty) \rangle.$$

The boundary conditions for $J = 0$, $\eta = 0$ have the form

$$\frac{\delta \langle S(\infty) \rangle}{\delta \bar{\eta}(p)} = i \langle \psi(p) \rangle; \tag{A2}$$

$$\frac{\delta \langle S(\infty) \rangle}{\delta \eta(p)} = - i \langle \psi(p) \rangle;$$

$$\frac{\delta \langle S(\infty) \rangle}{\delta J_\mu} = i \langle A_\mu(p) \rangle.$$

It follows from Eqs. (A1) and (A2) that

$$\langle S(\infty) \rangle = \exp\Big[i \int \Big\{ \bar{\eta}(p) \frac{1}{\hat{\mathrm{p}} + m} \eta(p) \tag{A3}$$

$$+ \frac{1}{2} J_\mu(p) \frac{1}{p^2} J_\mu(p) + J_\mu(p) \langle A_\mu(p) \rangle$$

$$+ \langle \psi(p) \rangle \eta(p) + \eta(p) \langle \psi(p) \rangle \Big\} d^4p \Big].$$

In particular, when an average is taken over the vacuum-vacuum states, in accordance with Eq. (38a) we get

$$(\Psi_0^*, S(\infty) \Psi_0) = \exp\Big[i \int \Big\{ \bar{\eta}(p) \frac{1}{\hat{\mathrm{p}} + m} \eta(p) \tag{A4}$$

$$+ \frac{1}{2} J_\mu(p) \frac{1}{p^2} J_\mu(p) \Big\} d^4p \Big].$$

Finding a solution for S does not present much difficulty, even in the case of the presence of interaction. The solution is carried out by means of the theory of excitation, assuming a solution in a power series in the charge. In this case, Eq. (A4) plays the role of the zeroth approximation.

b) *The Theory of Ferry.* In terms of the functional derivatives with respect to $< A_\mu >$, the theorem of Ferry can be formulated in the following manner: in the absence of external sources ($J = 0$, $\eta = 0$), the odd functional derivatives with respect to $< A_\mu >$ from the Green's function of the photons $D_{\mu\nu}$ are equal to zero. In fact, taking into account the charge symmetry of the theory, as is not difficult to show, the polarization operator $P_{\nu\mu}$ [see Eq. (51)]can be written in the form:

$$P_{\mu\nu}(p, k_1) \tag{A5}$$

$$= \frac{e^2}{2(2\pi)^4 i} \operatorname{Sp} \Big\{ \gamma_\mu \int \frac{\delta}{\delta e \langle A_\nu(k_1) \rangle} [G(p + k, k)$$

$$- \tilde{G}(p + k, k)] \Big\} d^4p.$$

where G is the Green's function of the charge-coupled equation. Here G is defined by an equation analogous to the equation for G [see Eq. (47)] only with this difference, that the charges e are taken with opposite sign. If we take the solution for G in the form of a functional series in $<A_\mu>$:

$$G(p, k) = \sum_{n=0}^{\infty} \int e^n G^{(n)}_{\mu_1 \dots \mu_n}(p, k, s_1 \dots s_n) \tag{A6}$$

$$\times \langle A_\mu(s_1) \rangle \dots \langle A_{\mu_n}(s_n) \rangle d^4s_1 \dots d^4s_n,$$

then the solution for $\tilde{G}(p, k)$ will have the form

$$\tilde{G}(p, k) = \sum_{n=0}^{\infty} \int (-e)^n G^{(n)}_{\mu_1 \dots \mu_n}(p, k, s_1 \dots s_n) \tag{A7}$$

$$\times \langle A_{\mu_1}(s_1) \rangle \dots \langle A_{\mu_n}(s_n) \rangle d^4s_1 \dots d^4s_n.$$

From Eqs. (A5), (A6) and (A7) (and taking into account that for $J=0$, $\dot{\eta}=0$), we find that the odd functional derivatives of $P_{\mu\nu}$ with respect to $\langle A_{\mu}\rangle$ are equal to zero for $J=0$.

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Bremsstrahlung from the Collisions of π Mesons with Nucleons *

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1. BREMSSTRAHLUNG from the scattering of mesons by nucleons, arising as a result of the coulomb interaction, was investigated in reference 2; yet, because of the strong interaction of mesons with nucleons, the principal role is played by bremsstrahlung from the nuclear force. Consequently, in the present work the cross section is calculated for the process of radiation of a γ-quantum in the scattering of a pseudoscalar π-meson from a nucleon by the nuclear force, both for pseudoscalar and for pseudovector coupling of the meson to the nucleon. The calculation is carried out to the third order in perturbation theory with the nonrelativistic approximation for the nucleon. Resulting formulas for the cross section of the process $\pi^- + p \to \pi^- + p + \gamma$ are obtained separately for the cases of pseudoscalar and pseudovector coupling.

2. The differential cross section in the coordinate system where the nucleon is originally at rest, for the case of pseudoscalar coupling, has the following form (the quantity $\hbar$ is everywhere designated by h):

$$d\sigma = \sigma_0 \frac{dE_\gamma}{E_\gamma} d\Omega_1 d\Omega_\gamma = \frac{1}{4\pi^2}\left(\frac{\mu}{2m}\right)^2 \frac{e^2}{hc}\left(\frac{h}{\mu c}\right)^2 \left(\frac{g^2}{hc}\right)^2 \tag{1}$$

$$\times \frac{k_1}{k_0}\frac{dE_\gamma}{E_\gamma}\left\{\frac{\beta_0^2 - (\vec{\beta}_0 \mathbf{n}_\gamma)^2}{(1-\beta_0\mathbf{n}_\gamma)^2} + \frac{\beta_1^2 - (\vec{\beta}_1\mathbf{n}_\gamma)^2}{(1-\beta_1\mathbf{n}_\gamma)^2}\right.$$

$$\left. - 2\frac{\vec{\beta}_0\vec{\beta}_1 - (\vec{\beta}_0\mathbf{n}_\gamma)(\vec{\beta}_1\mathbf{n}_\gamma)}{(1-\vec{\beta}_0\mathbf{n}_\gamma)(1-\vec{\beta}_1\mathbf{n}_\gamma)}\right\} d\Omega_1 d\Omega_\gamma,$$

where

$$\beta_0 = \frac{v_0}{c} = \frac{chk_0}{E_0}; \quad \beta_1 = \frac{v_1}{c} = \frac{chk_1}{E_1}; \quad \mathbf{n}_\gamma = \frac{\mathbf{k}_\gamma}{|\mathbf{k}_\gamma|};$$

and where $h\mathbf{k}_0$, E_0; $h\mathbf{k}_1$, E_1 are the momentum and total energy of the meson before and after the collision; $\hbar\mathbf{k}_\gamma, E_\gamma$ are the momentum and energy of the γ-quantum, m the mass of the nucleon and μ the mass of the π-meson.

For the investigation of the angular distribution of the γ-quanta, Eq. (1) can be rewritten in the form

$$d\sigma = \sigma_0\frac{dE_\gamma}{E_\gamma} d\Omega_1 d\Omega_\gamma = \frac{1}{4\pi^2}\left(\frac{\mu}{2m}\right)^2\left(\frac{g^2}{hc}\right)^2\left(\frac{h}{\mu c}\right)^2\frac{l^2}{hc} \tag{2}$$

$$\times\frac{k_1}{k_0}\frac{dE_\gamma}{E_\gamma}\left(\frac{\vec{\beta}_0}{1-\vec{\beta}_0\mathbf{n}_\gamma} - \frac{\vec{\beta}_1}{1-\vec{\beta}_1\mathbf{n}_\gamma}\right)^2 \sin^2\theta\, d\Omega_1 d\Omega_\gamma,$$

where θ is the angle between the vector $\left(\frac{\vec{\beta}_0}{1-\vec{\beta}_0\mathbf{n}_\gamma} - \frac{\vec{\beta}_1}{1-\vec{\beta}_1\mathbf{n}_\gamma}\right)$ and $\mathbf{k}_\gamma$. In the nonrelativistic approximation for the meson this formula has the form

$$d\sigma = \sigma_0\frac{dE_\gamma}{E_\gamma} d\Omega_1 d\Omega_\gamma \tag{2'}$$

$$= \frac{1}{4\pi^2}\left(\frac{\mu}{2m}\right)^2\left(\frac{g^2}{hc}\right)^2\frac{e^2}{hc}\left(\frac{h}{\mu c}\right)^2\frac{k_1}{k_0}(\vec{\beta}_0 - \vec{\beta}_1)^2$$

$$\times \sin^2\theta\frac{dE_\gamma}{E_\gamma} d\Omega_1 d\Omega_\gamma.$$

The angular distribution of the radiation is similar to that of a dipole vibrating along an axis which coincides with the vector $\mathbf{v}_0 - \mathbf{v}_1$.

The angular distribution of the γ-quanta, disregarding the angle of the scattered π-meson, is given by the formula

$$\frac{dE_\gamma}{E_\gamma} d\Omega_\gamma \int \sigma_0 d\Omega_1 = \frac{1}{\pi}\left(\frac{\mu}{2m}\right)^2\frac{e^2}{hc}\left(\frac{g^2}{hc}\right)^2\left(\frac{h}{\mu c}\right)^2 \tag{3}$$

$$\times\frac{k_1}{k_0}\frac{dE_\gamma}{E_\gamma} d\Omega_\gamma\left\{\frac{\beta_0^2\sin^2\vartheta}{(1-\beta_0\cos\vartheta)^2} + \frac{1}{\beta_1}\ln\frac{1+\beta_1}{1-\beta_1} - 2\right\},$$

where ϑ is the angle between the vectors $\mathbf{k}_0$ and $\mathbf{k}_1$. In nonrelativistic approximation for the meson this formula takes the form

$$\frac{dE_\gamma}{E_\gamma} d\Omega_\gamma \int \sigma_0 d\Omega_1 = \frac{1}{\pi}\left(\frac{\mu}{2m}\right)^2\frac{e^2}{hc}\left(\frac{g^2}{hc}\right)^2 \tag{3'}$$

$$\times \left(\frac{h}{\mu c}\right)^2\frac{k_1}{k_0}\frac{dE_\gamma}{E_\gamma}\left\{\beta_0^2\sin^2\vartheta + \frac{2}{3}\beta_1^2\right\}.$$

In this case of nonrelativistic energy for the meson, the majority of the γ-quanta emerge perpendicular to the direction of the incident mesons and with increasing energy up to this maximum deviation from the direction of the incident meson. The energy distribution of the γ-quanta has the characteristic formula

$$\frac{dE_\gamma}{E_\gamma}\int \sigma_0 d\Omega_1 d\Omega_\gamma \tag{4}$$

$$= 4\left(\frac{\mu}{2m}\right)^2\left(\frac{g^2}{hc}\right)^2\frac{e^2}{hc}\left(\frac{h}{\mu c}\right)^2\frac{k_1}{k_0}\frac{dE_\gamma}{E_\gamma}$$

$$\times\left\{\frac{1}{\beta_0}\ln\frac{1+\beta_0}{1-\beta_0}+\frac{1}{\beta_1}\ln\frac{1+\beta_1}{1-\beta_1}-4\right\}.$$

In the nonrelativistic approximation for the meson this expression takes the simpler form

$$\frac{dE_\gamma}{E_\gamma}\int \sigma_0 d\Omega_1 d\Omega_\gamma \tag{4'}$$

$$= \frac{8}{3}\left(\frac{\mu}{2m}\right)^2\left(\frac{h}{\mu c}\right)^2\frac{e^2}{hc}\left(\frac{g^2}{hc}\right)^2\frac{dE_\gamma}{E_\gamma}(\beta_0^2+\beta_1^2).$$

The total cross section is obtained by integrating over a very small energy interval δ_E about the kinetic energy of the incident meson $T = E_0 - \mu c^2$. For the case $E_0 \gg \mu c^2$ the total cross section takes the form:

$$\sigma = \int\frac{dE_\gamma}{E_\gamma}\sigma_0 d\Omega_1 d\Omega_\gamma \tag{5}$$

$$= 16\left(\frac{\mu}{2m}\right)^2\left(\frac{g^2}{hc}\right)^2\frac{e^2}{hc}\left(\frac{h}{\mu c}\right)^2\ln\frac{2E_0}{\mu c^2}\ln\frac{E_0}{\delta_E};$$

in the case of nonrelativistic energy for the meson

$$\sigma = \int\frac{dE_\gamma}{E_\gamma}\sigma_0 d\Omega_1 d\Omega_\gamma \tag{5'}$$

$$= 8\left(\frac{\mu}{2m}\right)^2\left(\frac{g^2}{hc}\right)^2\frac{e^2}{hc}\left(\frac{h}{\mu c}\right)^2\frac{2T_0}{\mu c^2}\left[\ln\frac{2T_0}{\delta_E}-\frac{5}{3}\right],$$

where $T_0 = (hk_0)^2/2\mu$, that is, the energy which the meson under consideration would have nonrelativistically. The total cross section increases as the first power of this kinetic energy, and at extreme relativistic energies the total cross section increases as the square of the logarithm of this energy.

3. In the case of pseudovector coupling the differential cross section takes the following form

$$d\sigma = \sigma_0\frac{dE_\gamma}{\mu c^2}d\Omega_1 d\Omega_\gamma \tag{6}$$

$$= \frac{1}{4\pi^2}\frac{e^2}{hc}\left(\frac{f^2}{hc}\right)^2\left(\frac{h}{\mu c}\right)^2\frac{k_1}{k_0}\frac{E_\gamma dE_\gamma}{(\mu c^2)^2}d\Omega_1 d\Omega_\gamma$$

$$\times\{((\vec\beta_0-\vec\beta_1)^2+\tfrac{3}{4}(\vec\beta_0\vec\beta_1-(\vec\beta_0\mathbf{n}_\gamma)(\vec\beta_1\mathbf{n}_\gamma))$$

$$+2\frac{\beta_1^2-(\vec\beta_1\mathbf{n}_\gamma)^2}{1-\vec\beta_1\mathbf{n}_\gamma}\left(\vec\beta_0\mathbf{n}_\gamma+\frac{E_0^2}{2mc^2E_\gamma}\right)$$

$$+2\frac{\beta_0^2-(\vec\beta_0\mathbf{n}_\gamma)}{1-\vec\beta_0\mathbf{n}_\gamma}\left(\vec\beta_1\mathbf{n}_\gamma-\frac{E_1^2}{2mc^2E_\gamma}\right)$$

$$+\frac{\beta_0^2-(\vec\beta_0\mathbf{n}_\gamma)^2}{(1-\vec\beta_0\mathbf{n}_\gamma)^2}\left(\beta_1^2\left(\frac{k_0}{k_\gamma}\right)^2+\beta_1^2-2\beta_1\frac{E_0}{E_\gamma}\right.$$

$$\left.+\frac{E_1^4}{E_\gamma^2(2mc^2)^2}+\frac{(ch)^2(\mathbf{k}_1\mathbf{k}_0-\mathbf{k}_\gamma)E_1}{mc^2E_\gamma}\right)$$

$$+\frac{\beta_1^2-(\vec\beta_1\mathbf{n}_\gamma)^2}{(1-\vec\beta_1\mathbf{n}_\gamma)^2}\left(\beta_0^2\left(\frac{k_1}{k_\gamma}\right)^2+\beta_0^2+2\beta_0^2\frac{E_1}{E_\gamma}\right.$$

$$\left.+\frac{E_0^4}{E_\gamma^2(2mc^2)^2}+2\frac{(ch)^2(\mathbf{k}_0,\mathbf{k}_1+\mathbf{k}_\gamma)}{2mc^2E_\gamma^2}E_0\right)$$

$$-2\frac{\vec\beta_0\vec\beta_1-(\vec\beta_0\mathbf{n}_\gamma)(\vec\beta_1\mathbf{n}_\gamma)}{(1-\vec\beta_0\mathbf{n}_\gamma)(1-\vec\beta_1\mathbf{n}_\gamma)}\left[\beta_0^2\beta_1^2\frac{E_0E_1}{E_\gamma^2}\right.$$

$$-\beta_1^2\frac{E_1}{E_\gamma}+\beta_0^2\frac{E_0}{E_\gamma}+\frac{E_0^2E_1^2}{E_\gamma^2(2mc^2)^2}+\frac{E_1^2}{2mc^2E_\gamma}-\frac{E_0^2}{2mc^2E_\gamma}$$

$$+(\vec\beta_1\mathbf{n}_\gamma)\left(1-\frac{E_0^2+E_1^2}{2mc^2E_\gamma}\right)+(\vec\beta_0\mathbf{n}_\gamma)\left(-1+\frac{E_0^2+E_1^2}{2mc^2E_\gamma}\right)$$

$$\left.+(\vec\beta_0\vec\beta_1)\left(1+\frac{E_0^3+E_1^3}{2mc^2E_\gamma^2}\right)\right]\Bigg\}.$$

With further analysis this formula, carried out in the range of energies which satisfy the inequality $1.1\mu c^2 < E_0 \ll mc^2$, takes the form

$$d\sigma = \sigma_0\frac{dE_\gamma}{\mu c^2}d\Omega_1 d\Omega_\gamma \tag{6'}$$

$$= \frac{1}{4\pi^2}\left(\frac{f^2}{hc}\right)^2\frac{e^2}{hc}\left(\frac{h}{\mu c}\right)^2\frac{E_\gamma dE_\gamma}{(\mu c^2)^2}d\Omega_1 d\Omega_\gamma$$

$$\times\left\{(\vec\beta_0-\vec\beta_1)^2+\frac{4\beta_0^2\quad\beta_1^2}{(\beta_0^2-\beta_1^2)^2}[(\beta_0^2-(\vec\beta_1\mathbf{n}_\gamma)^2)\right.$$

$$+(\beta_1^2-(\vec\beta_1\mathbf{n}_\gamma)^2)-2(\vec\beta_0\vec\beta_1-(\vec\beta_0\mathbf{n}_\gamma)(\vec\beta_1\mathbf{n}_\gamma))]$$

$$\left.+\frac{4}{\beta_0^2-\beta_1^2}[\beta_1^2(\vec\beta_0\mathbf{n}_\gamma)^2-\beta_0^2(\vec\beta_1\mathbf{n}_\gamma)^2]\right\}.$$

The angular distribution of the γ-quanta, disregarding the angle of the scattered π-meson, has the characteristic form

$$\frac{dE_\gamma}{\mu c^2}d\Omega_\gamma\int\sigma_0 d\Omega_1 \tag{7}$$

$$= \frac{1}{\pi}\frac{e^2}{hc}\left(\frac{f^2}{hc}\right)^2\left(\frac{h}{\mu c}\right)^2\frac{k_1}{k_0}\frac{dE_\gamma E_\gamma}{(\mu c^2)^2}d\Omega_\gamma$$

$$\times\left\{\beta_0^2+\beta_1^2+4\,\frac{\beta_0^2\beta_1^2}{(\beta_0^2-\beta_1^2)^2}\left[\frac{\beta_0^2\sin^2\vartheta_0}{(1-\beta_0\cos\vartheta_0)^2}+\frac{2}{3}\,\beta_1^2\right].\right.$$

$$\left.+\,4\,\frac{\beta_0^2\beta_1^2}{\beta_0^2-\beta_1^2}\left(\cos^2\vartheta_0-\frac{1}{3}\right)\right\}.$$

The energy distribution of the γ-quanta is obtained after integrating over angles :

$$\frac{dE_\gamma}{\mu c^2}\int\sigma_0 d\Omega_\gamma\, d\Omega_1 \tag{8}$$

$$=4\,\frac{e^2}{hc}\left(\frac{f^2}{hc}\right)^2\left(\frac{h}{\mu c}\right)^2\frac{k_1}{k_0}\,\frac{E_\gamma dE_\gamma}{(\mu c^2)^2}$$

$$\times\ \left\{\beta_0^2+\beta_1^2+\frac{2}{3}\,\beta_0^2\beta_1^2\,\frac{1}{E_\gamma^2}\left[(chk_1)^2+(chk_0)^2\right]\right\}.$$

The total cross section for the case $\mu c^2 \ll E_0 \ll mc^2$ takes the form

$$\sigma=\int\sigma_0 d\Omega_\gamma d\Omega_1\,\frac{dE_\gamma}{\mu c^2} \tag{9}$$

$$=2\,\frac{e^2}{hc}\left(\frac{f^2}{hc}\right)^2\left(\frac{h}{\mu c}\right)^2\frac{4}{3}\left(\frac{E_0}{\mu c^2}\right)^2\ln\frac{2E_0}{\mu c^2}\ln\frac{E_0}{\delta_E}\,.$$

Note that the cross section for the process $\pi^+ + p \rightarrow \pi^+ + p + \gamma$ differs somewhat from the cross section for the process $\pi^- + p \rightarrow \pi^- + p + \gamma$ in the case of pseudovector coupling.

4. The cross section obtained by us, $d\sigma_T$, for bremsstrahlung from a nonrelativistic energy meson and with $E_\gamma \ll E_0$, together with the cross section for elastic scattering of π-mesons by nucleons derived in reference 5, gives the most general relationship

$$d\sigma_T=\frac{2}{3\pi}\,\frac{e^2}{hc}\,\frac{(v_0-v_1)^2}{c^2}\,\frac{dE_\gamma}{E_\gamma}\,d\sigma_{\pi\text{-}\mathrm{el.}} \tag{10}$$

An analogous relation has been obtained earlier in references 3-5.

The total cross section for the process of bremsstrahlung from the scattering of π-mesons by nucleons for both types of coupling of the meson to the nucleon has the order of magnitue of 10^{-28} cm^2 for $E_0 \sim 2\mu c^2$, for $\frac{f^2}{hc}=\frac{\mu}{2m}\frac{g^2}{hc}\approx\frac{1}{6}$ and for the energy interval $\delta_E = 1$ mev.

In conclusion, I wish to thank Professors I. I. Pomeranchuk and I. M. Shmushkevich for assistance and guidance in this work.

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* This communication is based on work completed in 1952 at the Institute for Nuclear Problems of the Academy of Sciences, USSR.

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The Role of Isobaric States of Nucleons in Meson Creation

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As is known[1], there is experimental evidence that the isobaric states of the nucleons play an important role in the process of meson production in nucleon-nucleon collisions. A series of authors have taken the point of view that the production of pions proceeds entirely through an isobaric state[2-4]. However, it has not been excluded that direct production plays a significant role in such processes.

Let us examine the experiments on production of charged pions in the reaction $Be^9 + p$ [2]. In these experiments the ratio $\rho = \pi^+/\pi^-$ (π^+ and π^- indicatingthe number of π^+ and π^- mesons formed) is equal to 6 at an energy of 1 bev, and to 1.8 at 2.3 bev. In the paper of Peaslee[2] there is a discussion of the possibility of explaining such a dependence of ρ on energy by assuming that the production of pions proceeds only through an isobaric state. At an energy of 2.3 bev, in order to calculate ρ, one must evaluate matrix elements or else make definite assumptions about them[2]. At 1 bev, however, there is not enough energy to produce real excitation of two isobaric states and one must therefore assume that the production of pions goes only by the excitation of one isobaric state. Then, assuming only the hypothesis of charge independence, it is easy to calculate ρ. The result obtained by Peaslee[2] for ρ is equal to 6, agreeing well with experiment. However, in this work an error was made, as pointed out by the author himself in a following paper[3]. The corrected value of ρ at 1 bev is equal to 9, no longer agreeing with the experiment.

We have carried out a calculation of ρ on a statistical theory[4]. In the calculation we have included both the production of pions via the isobaric state and direct production. The relative importance of the two processes is determined only by statistical weights. It is assumed that at these energies in the reaction $\text{Be}^9 + p$, the incoming proton interacts with each of the nucleons of the nucleus as if it were free.

Let us examine first the situation when the energy of the incoming protons is equal to 1 bev. Then only the following processes are important: NN, NN', $NN\pi$ (N – nucleon, N' – isobaric state). These correspond to elastic scattering, excitation of one isobar and direct production of one meson. These processes are easily calculated by the formulas given by Belinfante[4]. Simple calculations yield 5 for the value of ρ. At a proton energy of 2.3 bev, besides processes producing 1 or 2 mesons, 3 meson production has some importance. This corresponds to the processes $N'N'\pi$, $NN'\pi\pi$ and $NN\pi\pi\pi$. Evaluation indicates, however, that these reactions occur in only 4 to 5% of all collisions. The calculation of ρ at 2.3 bev yields 1.8, agreeing exactly with the experimental result.

We have calculated, likewise, values of ρ for intermediate energies. These are as follows:

E	ρ
1	5
1.46	2.7
1.75	2
2.3	1.8

Thus, both the value of ρ and its dependence on energy agree with the calculation according to the statistical theory, in which is included both direct production of π mesons and their creation via isobaric states. Inclusion in the statistical theory only of direct production of mesons given at 1 bev, $\rho = 3.5$ and at 2.3 bev, $\rho = 2.7$.

In conclusion, I am grateful to Professor C. E. Belen'kii for cricticism of this work and for valuable suggestions.

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Numerical Values of the Constant of the Triplet Beta-Interaction

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THE ratio of the constants of the two elementary interactions leading to allowed beta-transitions can be estimated from experimental data[1,2] which have appeared in recent times. We shall employ nonrelativistic terminology and call these interactions singlet (Fermi selection rules) and triplet (Gamow-Teller rules).

In the general case of a mixed transition the ft value found from experimental data is connected with the interaction constants by the equation

$$ft\left[\frac{M_0^2}{A_0} + \frac{M_1^2}{A_1}\right] = 1, \tag{1}$$

where M_0 and M_1 are the nuclear matrix elements for the singlet and triplet interactions; the universal times A_0 and A_1 are inversely proportional to the squares of the interaction constants G:

$$A = \frac{2\pi^3 \ln 2\, \hbar^7}{m_0^5 c^4 G^2} = \frac{1.2\cdot 10^{-94}}{G^2}. \tag{2}$$

For a pure process $A = ftM^2$.

The nuclear matrix elements M_0 for the singlet interaction are found theoretically. For this, no other assumptions are required besides that of charge invariance. The latter is violated only at the expense of a distortion of the nucleon wave functions by the Coulomb field. For the triplet interaction the nuclear matrix elements M_1 can be found theoretically only for pure states.

Because of the indicated reasons only the singlet constant can be found directly from the ft value for a pure process. For the singlet interaction it was found directly from the decay of O^{14} that $A_0 = ftM_0^2 = 6550 \pm 150$ sec.

For processes going by a pure triplet interaction the matrix elements are unknown. Therefore, one has to turn to mixed transitions for an estimate of the triplet constant. An exact value of the matrix element M_1 is known only for the free neutron, whose lifetime has been measured with very low accuracy. But we can give an accurate upper limit of the quantity A_1 in those transitions for which an upper limit of M_1^2 is known. The beta decay of triton yields the extreme of such estimates. On the basis of detailed calculations[3] one can assume that here M_1 cannot be greater than three. The value $ft = 1014 \pm 20$ sec has been obtained from very precise measurements[2].

From this we obtain that $A_1 \leq 3600$ sec. This upper limit practically coincides with the very lower limit which can be derived from the lifetime of the free neutron measured by Robson[4] with a quoted error of ±18%. Evidently, the errors of measurement specified by Robson[4] are to be regarded as probable errors in the statistical sense, but not as outer limits of the error of measurement. Apparently, the actual error of these measurments[4] is larger, inasmuch as M_1^2 for triton must be less than three.

Thus, it turns out that the lifetime of the free neutron, at the precision with which it is known at present, yields practically nothing for the determination of the constant of beta-decay.

A lower limit for A_1 can at present be obtained only from astrophysical data. Among all the possible models of the sun, the pure hydrogen-helium model gives the lowest rate of heat production. In addition to this, the total heat production is provided by the hydrogen cycle, whose rate is determined by the triplet beta-process $H^1+H^2=H^2+e^2+\nu$.

This is the only beta-transition for which the matrix element can be calculated accurately from theory, which has been done by Frieman and Motz[5] and also by Salpeter[6]. The observed energy production of the sun is provided by the hydrogen cycle alone with $A_1 = 2060$ sec, which can be regarded as a lower limit of the quantity A_1.

Thus, the value of A_1 must lie inside the limits

$$3600 \geq A_1 \geq 2060 \text{ sec},$$

and the ratio of the constants $R = G_1^2/G_0^2 = A_0/A_1$ is between the limits $3.18 \geq R \geq 1.82$.

All theories that require equality of the constants[7,8] are thus completely eliminated.

It should be emphasized that the present discussion is not about statistical errors but about upper and lower limits. Hence, the half-life of the free neutron must lie within the limits $600 \geq t \geq 370$ sec.

We can likewise estimate limits within which the values of M_1^2 for the simplest beta-transitions must lie. By substituting the experimental ft values, the known values of M_0^2 and A_0, and the limiting values of A_1 into (1), we obtain

	ft	M_0^2		
$He^6 - Li^6$	815	0	$4.4 \geqslant M_1^2 \geqslant 2.5$	(6)
$H^3 - He^3$	1014	1	$3.0 \geqslant M_1^2 \geqslant 1.7$	(3)
$N^{13} - C^{13}$	4700	1	$0.22 \geqslant M_1^2 \geqslant 0.12$	$(^1/_3)$
$O^{15} - N^{15}$	3750	1	$0.41 \geqslant M_1^2 \geqslant 0.25$	$(^1/_3)$
$F^{17} - O^{17}$	2420	1	$1.0 \geqslant M_1^2 \geqslant 0.52$	$(^7/_5)$
$Be^7 - Li^7$	2547	1	$0.87 \geqslant M_1^2 \geqslant 0.49$	$(^5/_3)$
$Be^7 - Li^{7*}$	3590	0	$1.00 \geqslant M_1^2 \geqslant 0.56$	$(^4/_3)$

For comparison, the reduced values of M_1^2 for the nearest pure states are given in parentheses.

The cited limits do not include the experimental error, which, however, is small for these transitions: in all cases the error in ft does not exceed 3%.

For N^{13} and F^{17} our estimates of M_1^2 differ markedly from the estimates cited in the literature[9,10], which were derived from magnetic moments.

I am grateful to Ia. B. Zel'dovich and Iu. A. Romanov for valuable comments.

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Consideration of the Nuclear Quadrupole Moment in Electron Scattering

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AT the present time many experiments have been carried out on the elastic scattering of electrons by nuclei. In this connection, it is of interest to estimate the influence on this effect of the nuclear quadrupole moment. The latter is taken into account in the present paper with the aid of the Born approximation according to the following considerations.

It was shown by Vachaspati[1] that, if account is taken of the first and second Born approximation, the results are in satisfactory agreement with exact scattering theory. Since the quadrupole moment forms only a correction to the interaction, it is appropriate to consider it in first approximation only.

Baranger[2] has given scattering curves obtained form the exact theory and from the Born approximation. It is evident from a comparison of these that, although they coincide only at certain points, they are close together, in order of magnitude, over a wide range of angles.

The discussion just given justifies the calculation that has been carried out. The effective cross section of the elastic scattering of an electron by a nucleus is written

$$d\sigma = \frac{1}{4\pi^2}\,\frac{E^2}{h^4c^4}\,|V_{ab}|^2\,dO, \tag{1}$$

where E is the energy of the scattered electron, V_{ab} is the matrix element of interaction, which has the form

$$V = Ze\varphi + \frac{1}{6}\sum_{i,k=1}^{3}\frac{\partial^2\varphi}{\partial x_i\partial x_k}D_{ik}. \tag{2}$$

here φ is the scalar potential[3]

$$\varphi = -\frac{4\pi ec}{q^2}\,a'^{*}\,a^0e^{i\mathbf{q}\cdot\mathbf{r}}, \quad \mathbf{q} = \frac{\mathbf{p}-\mathbf{p}_0}{h}; \tag{3}$$

a'^{*}, a^0 are spinors which characterize the state of the electron before and after the scattering, D_{ik} is the quadrupole moment tensor, given by[4]

$$D_{ik} = \frac{3eD_0}{2L(2L-1)}\left(\hat{I}_i\hat{I}_k + \hat{I}_k\hat{I}_i - \frac{2}{3}\delta_{ik}\hat{I}^2\right), \tag{4}$$

D_0 is the constant known as the nuclear quadrupole moment (dimensions = cm^2), I_i are the operators of the nuclear spin, L is the quantum number of the total nuclear spin.

If $qa \ll 1$, where a is the nuclear radius, then the scattering cross section, summed over the original states, has the form

$$d\sigma = \left[1 + \frac{q^4D_0^2}{30(2l+1)(2L-1)^2Z^2}\left(\frac{2}{3}L^2(2L+1)(L+1)^2 - 3\sum_0^L M^2\right)\right]d\sigma_M. \tag{5}$$

Here $d\sigma_M$ is the effective cross section of elastic electron scattering by a point nucleus, and is given by Mott's formula. The second term in the brackets gives the contribution brought about by the presence of the quadrupole moment. We estimate the order of magnitude of this contribution. For unit spin, $L = 1$,

$$d\sigma = \left[1 + \frac{q^4D_0^2}{18Z^2}\right]d\sigma_M. \tag{6}$$

If we select a scattering angle of 30°, and electronic energy 60 mev, then $q \sim 10^{12}\,\mathrm{cm}^{-1}$; in such a case the condition $qa \ll 1$ is satisfied. If D_0 is considered to be of the order of $10^{-24}\,\mathrm{cm}^2$, which is correct for heavy nuclei, then the correction amounts to

$$q^4D_0^2/18Z^2 = 1/18Z^2.$$

For the deuteron, $D_0 = 10^{-27}\,\mathrm{cm}^2$, and the second term is considerably smaller:

$$q^4D_0^2/18 = 1/(18\times 10^6).$$

Since there are at present some indications of the possibility of the creation of polarized nuclei, we give results of the calculation of some effective cross sections in this case. We assume that the nucleus, which has a spin L, is completely polarized, and that the projection of its spin moment in the direction of the polarizing field is also equal to L. The effective differential scattering cross section of the electron, when the projection of the nuclear spin does not change its value (transition $L - L$), has the form

$$d\sigma_{L-L} = \left(1 - \frac{D_0q^2P_2(\cos\theta)}{6Z}\right)^2 d\sigma_M. \tag{7}$$

Here $P_2(\cos\theta)$ is the Legendre polynomial, θ is the angle between the direction of the spin vector of the nucleus and the vector $\mathbf{q}$. It is evident that the effect of the nuclear quadrupole moment is determined by the term $D_0q^2/6Z$. If again $q = 10^{12}\,\mathrm{cm}^{-1}$, $D_0 = 10^{-24}\,\mathrm{cm}^2$, then we have $D_0q^2/6Z = 1/6Z$. For deuterons in this case, $D_0q^2/6 = 1/(6\times 10^3)$.

To take into account imcomplete polarization of the nucleus, we introduce formulas which correspond to the following possible transitions. Let the spin L have a projection M in the direction of the field. The effective scattering cross section of the electron for the case of the same final state of nuclear spin (transition $M - M$) is given by

$$d\sigma_{MM} = \left\{1 - \frac{[3M^2 - L(L+1)]\,q^2D_0}{6L(2L-1)Z}\right. \tag{8}$$

$$\times P_2(\cos\theta)\Big\}^2 d\sigma_M.$$

For the effective cross section, averaged over all directions of the vector **q**, we have

$$d\sigma_{MM} = \left\{1 + \frac{[3M^2 - L(L+1)]^2}{180\,L^2(2L-1)^2 Z^2} q^4 D_0^2\right\} d\sigma_M. \quad (9)$$

For the transition $(M, M+1)$ (initial spin state of the nucleus M, final, $M+1$) we have

$$d\sigma_{M,M+1} = \frac{q^4 D_0^2 (2M+1)^2 (L+M+1)(L-M)}{16L^2(2L-1)^2 Z^2} \quad (10)$$

$$- d\sigma_M (1 - \cos^2\theta)\cos^2\theta.$$

The effective cross section, averaged over all directions of the vector **q**, is given in the form

$$d\sigma_{M,M+1} = \frac{q^4 D_0^2 (2M+1)^2 (L+M+1)(L-M)}{120L^2(2L-1)^2 Z^2} d\sigma_M. \quad (11)$$

Finally, transitions $(M, M+2)$ are possible:

$$d\sigma_{M,M+2} = \frac{q^4 D_0^2 d\sigma_M}{64L^2(2L-1)^2 Z^2}(L+M+2) \quad (12)$$

$$\times (L-M-1)(L+M+1)(L-M)(1-\cos^2\theta)^2.$$

The expression, averaged over the directions **q**, has the following form:

$$d\sigma_{M,M+2} = \frac{q^4 D_0^2 (L+M+2)}{120L^2(2L-1)^2 Z^2} \quad (13)$$

$$\times (L-M-1)(L+M+1)(L-M)\, d\sigma_M.$$

In this case, when there is a certain distribution of the directions of the nuclear spin relative to the field, the effective differential cross section, averaged over the initial states and summed over the final, will have the form

$$d\sigma = \frac{1}{(2L+1)} \sum_M a_L(M)[d\sigma_{MM} +$$

$$\times\ d\sigma_{M,M+1} + d\sigma_{M,M+2}],$$

where $a_L(M)$ is the probability of finding the nuclear spin L with projection M before scattering, $d\sigma_{MM}$, $d\sigma_{M,M+1}$, $d\sigma_{M,M+2}$ are given by the corresponding formulas (9), (11) and (13). The transitions $(M, M-2)$ $(M, M-1)$ are not considered here, since they correspond to inelastic scattering in the case of polarized nuclei.

In conclusion, I thank Professor Ia. A. Smorodinskii for his valued comments.

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Nuclear Capture of a Negative Heavy Meson

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THERE has been found in type *R* photographic plates with an emulsion thickness of 300 μ exposed in the stratosphere, an event, the microphotograph of which is represented in the drawing. The visible track of particle 1 consists of 495 μ. From the change in ionization and scattering along the track it is obvious that particle 1 stopped at point *A*. From this point there start two tracks: one gray and one very short black track (~ 1 μ). The presence of the short black track is evidence for nuclear capture of the first particle, which, therefore, can be either a negative π meson or a heavier negative particle.

Particle 2 leaves the emulsion after traveling 674 μ. Its ionization is 3.2 ± 0.3 times minimum. From this it follows that particle 1 is heavier than a π meson since, even if it is assumed that particle 2 is a proton, then its energy must be ~ 200 mev. A proton of such energy cannot be formed upon nuclear capture of a π meson.

A direct determination of the mass of the secondary particle from its ionization and multiple scattering leads to a value of $(350 \pm 200)\, m_e$. It is more realistic to consider this particle as a π meson. Then its energy is ~ 30 mev.

Comparison of the multiple scattering and gap count along the track of the first particle with the range indicates that its mass lies between that of the π meson and the proton.

All this is interpretable as the nuclear capture of a stopped negative heavy meson. Rather striking is the exceptionally small energy release and the production of a π meson with energy ~ 30 mev, the same as upon decay of a Λ^0 particle.

In conclusion, the authors express their deep gratitude to I. M. Gramenitzky and M. I. Podgoretzky for discussions.

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Calculation of the Ultimate Thickness of the Emulsion Layer in the Investigation of Nuclear Processes with the Aid of the Photographic Method

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(August, 1955)

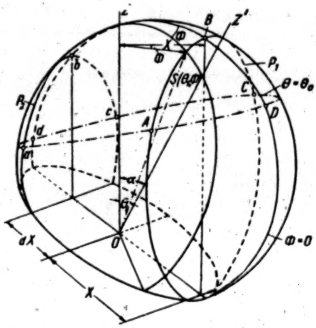
In a previous study[1] we obtained an analytical expression for the relation of the general number of those secondary particles which were formed during a nuclear reaction in the photoemulsion to the number of those secondary particles whose traces are to be found in their entirety within the emulsion layer. It was assumed thereby that the primary particles fall parallel to the emulsion layer. Below are presented analogous formulas for the more general case where the primary particles fall at a given angle α toward the surface of the emulsion layer.

In the following diagram is shown the reciprocal arrangement of the emulsion layer and its lateral surfaces P_1, P_2 of the surface $S = S(\vartheta)$ representing the geometrical locus of the ends of the traces of secondary particles which flew out at an angle ϑ toward the axis OZ', equal to the direction of incidence of the primary particles and of the cone K with its apex at the point where the secondary particles are formed, and its axis directed along the projection of the OZ'-axis on the plane P_0. The apex angle $2\theta_0$ of the cone as well as the incline α of its axis, are given by the conditions of the experiment.

We designate, as before[1], by σ and σ' the magnitudes which are proportional, respectively, to the full number of the secondary particles diffusing or scattering within cone K, and to the number of secondary particles whose traces, remaining within cone K, terminate outside the emulsion layer. The sought-for magnitude of the correction w is then determined by the formula[1] $w = 1 - \sigma'/\sigma$. If we designate by $f(\vartheta)$ the distribution of the intensity diffusion (scattering) of the secondary particles, then we obtain an expression for σ shown by the following formula:

$$\sigma = \int_0^{2\pi} d\Phi \int_0^{\theta_0} f(\vartheta) \sin\theta \, d\theta, \tag{1}$$

where



Reciprocal arrangement of the surface $S = S(\Theta, \Phi)$ and of the cone K in the emulsion layer.

$$\cos\vartheta = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos\Phi. \tag{2}$$

We now formulate the expressions for σ' applicable to the following three cases:

I. The thickness d of the emulsion layer is minute and satisfies the condition: $d < \delta = \min S \times (\theta_0, \Phi) \sin\theta_0$,

$$\frac{d}{2}\sigma' = \int_0^{\pi/2} F_1(\theta_0, \Phi)\, d\Phi - \int_{\pi/2}^{\pi} F_1(\theta_0, \Phi)\, d\Phi$$
$$- \int_0^{\psi_2} F_2(\theta_0, \Phi)\, d\Phi + \int_{\psi_2}^{\pi} F_3(\theta_0, \Phi)\, d\Phi$$
$$+ \int_0^{\psi_2} F_2(\bar{\theta}, \Phi)\, d\Phi - \int_{\psi_3}^{\pi} F_3(\theta', \Phi)\, d\Phi, \tag{3}$$

where the following designations are introduced:

$$F_1(\theta, \Phi) = \cos\Phi\,[M(\theta_0, \Phi) - M(0, \Phi)];$$

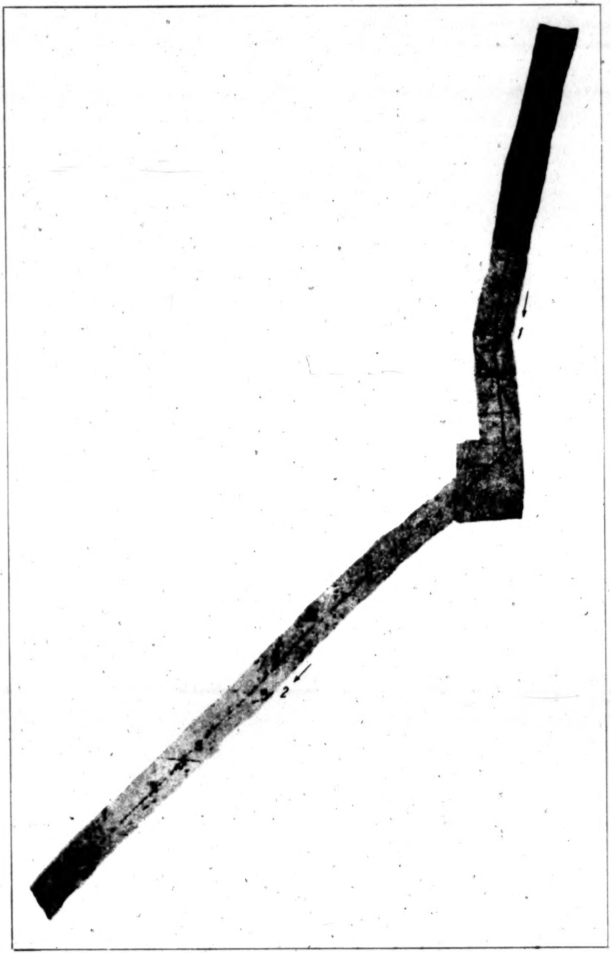
$$F_2(\theta, \Phi) = \{\cos\Phi \cdot M(\theta, \Phi) - d\,\Lambda(\theta, \Phi)\};$$

$$F_3(\theta, \Phi) = \{\cos\Phi\, M(\theta, \Phi) + d\, N(\theta, \Phi)\}.$$

In this case, the magnitudes $\psi_2, \psi_3, \bar{\theta}, \theta'$ are determined, respectively, from the following equations:

$$\cos\psi_2 = \frac{d}{S(\theta_0, \psi_2)\sin\theta_0}; \quad \cos\psi_3 = \frac{-d}{S(\theta_0, \psi_3)\sin\theta_0};$$

$$\sin\bar{\theta} = \frac{d}{S(\bar{\theta}, \Phi)\cos\Phi}; \quad \sin\theta' = \frac{-d}{S(\theta', \Phi)\cos\Phi}.$$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II. The thickness d of the emulsion layer satisfies the condition:

$$d > \Delta = \max S(\theta_0, \Phi)\sin\theta_0.$$

In this case, $\psi_2 = 0$ and $\psi_3 = \pi$ and the expression for σ' has the aspect presented by the next formula:

$$\frac{d}{2}\sigma' = \int_0^{\pi/2} F_1(\theta_0, \Phi)\,d\Phi + \int_{\pi}^{\pi/2} F_1(\theta_0, \Phi)\,d\Phi. \tag{4}$$

III. If the thickness d of the emulsion layer satisfies the condition $\delta < d < \Delta$, then we have either $\psi_2 = 0$; $\psi_3 < \pi$, or $\psi_3 = \pi$ and $\psi_2 > 0$; consequently, the corresponding integrations (quadratures) in Eq. (3) disappear.

In all the enumerated formulas, the functions $M(\theta, \Phi)$ and $N(\theta, \Phi)$ denote, respectively, the following equations:

$$M(\theta, \Phi) = \int f(\vartheta)\,S(\vartheta)\sin^2\theta\,d\theta;$$

$$N(\theta, \Phi) = \int f(\vartheta)\sin\theta\,d\theta.$$

If we approximate $f(\vartheta)$ and $S(\vartheta)$ with polynomials with regard to $\cos\vartheta$[2,3], then the calculation of the first four integrations (quadratures) in Eq. (3) can be performed easily and exactly. The last two integrations (quadratures), however, can be calculated by means of any formula of mechanical quadratures.

In this way formula (3) [accordingly, Eq. (4) in the case II], together with formula (1), fully solve the problem of finding the correction $w = 1 - (\sigma'/\sigma)$.

From the cited formulas it is easy to obtain the correction w for the case which was examined earlier[1]. For this purpose we posit $\alpha = 0$, then $\vartheta \equiv \theta$ and $S(\vartheta) \equiv S(\theta)$. The expressions for σ' assume, therefore, the following aspect. In case I:

$$\frac{d}{4}\sigma' = M(\theta_0) - M(0) + d\,\psi_3 N(\theta_0) \tag{5}$$

$$- M(\theta_0)\sin\psi_3 + \int_0^{\psi_3} F(\overline{\theta})\,d\Phi.$$

Accordingly, in case II, we obtain:

$$\sigma' = \frac{4}{d}\left[M(\theta_0) - M(0)\right]. \tag{6}$$

Formulas (5) and (6) are simpler for practical calculations than the formulas cited earlier[1].

Let us remark that the choice of the layer thickness d and of the angle θ_0 permits us to reduce the majority of practical problems to case II. If, at the same time, $f(\vartheta)$ and $S(\vartheta)$ are given in the form of polynomials with regard to $\cos\vartheta$, then the computation of the correction w according to formulas (1) and (4) does not present any difficulties whatever.

I wish to express here my appreciation to Dr. Westmeier for his formulation of the problem and to A. Benet for a certain preliminary analysis.

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Translated by M. I. Weinreich
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An Elementary Derivation of the Formula for the Electromagnetic Energy in a Dispersive Medium

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LET us suppose that a substance of dielectric and magnetic susceptibility $\epsilon(\omega)$ and $\mu(\omega)$, respectively, fills a parallel plate condenser of capacity $C(\omega) = \epsilon(\omega)C_0$ and a thin solenoid of inductance $L(\omega) = \mu(\omega)L_0$, thus forming an oscillatory circuit. The free space values C_0 and L_0 are so chosen that the frequency ω is the natural frequency of the system given by: $\omega^2 = [L(\omega)C(\omega)]^{-1}$. Let us suppose that at $t < 0$ undamped oscillations take place in the circuit. The current through the solenoid, $Ie^{i\omega t}$, and the potential across the condenser, $Ve^{i\omega t}$, due to these oscillations satisfy the well-known condition $V = -i\sqrt{L/C}\,I$.

At $t = 0$ let us insert into the circuit a vanishingly small resistance R; then, at $t > 0$, the oscillations will have a complex frequency $\tilde{\omega}$, determined by the relation:

$$\tilde{\omega}L(\tilde{\omega}) - 1/\tilde{\omega}C(\tilde{\omega}) = iR.$$

It is easy to see that for the case of $R \to 0$, the solution of the above equation is $\tilde{\omega} = \omega + i\delta$, where δ and R are related by the equation:

$$\frac{R}{\delta} = \frac{d}{d\omega}(\omega L) + \frac{1}{\omega^2 C^2}\frac{d}{d\omega}(\omega C) \tag{1}$$

$$= \frac{d}{d\omega}(\omega L) + \frac{L}{C}\frac{d}{d\omega}(\omega C).$$

Clearly, the initial energy of the circuit, W, is equal to the total dissipated heat energy.

$$W = \int_0^\infty \frac{1}{2} R\,|I|^2 e^{-2\delta t} dt = \frac{R}{4\delta}\,|I|^2.$$

Substituting Eq. (1) into the above relation, and noting that $\frac{L}{C}|I|^2 = |V|^2$, we obtain:

$$W = \frac{1}{4}\left[\,|I|^2 \frac{d}{d\omega}(\omega L) + |V|^2 \frac{d}{d\omega}(\omega C)\right] \tag{2}$$

$$= \frac{1}{4} L_0\,|I|^2 \frac{d}{d\omega}(\omega\mu) + \frac{1}{4} C_0\,|V|^2 \frac{d}{d\omega}(\omega\varepsilon).$$

However, $L_0|I|^2 = \frac{1}{4\pi}\,|H|^2\,\tau_m$, $C_0|V|^2 = \frac{1}{4\pi}\,|E|^2\,\tau_e$, where H is the amplitude of the magnetic vector, τ_m is the volume of the solenoid, E is the amplitude of the electric vector, and τ_e is the volume of the capacitor. Hence, Eq. (2) can be written as follows:

$$W = \frac{|H|^2}{16\pi} \frac{d}{d\omega}(\omega\mu)\,\tau_m + \frac{|E|^2}{16\pi} \frac{d}{d\omega}(\omega\varepsilon)\,\tau_e. \tag{3}$$

From this we can conclude immediately, that the average electric and magnetic energy densities of a sinusoidal field in a dispersive medium are given by:

$$w_e = \frac{|E|^2}{16\pi} \frac{d}{d\omega}(\varepsilon\omega);\quad w_m = \frac{|H|^2}{16\pi} \frac{d}{d\omega}(\mu\omega). \tag{4}$$

These relations were obtained in references 1 and 2 by means of the Fourier integral method.

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184

The Second Viscosity of Monatomic Gases

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IT is well-known that monatomic gases, which obey Boltzmann statistics and which possess an energy spectrum of the form

$$\varepsilon = p^2/2m, \tag{1}$$

do not have a second viscosity[1]. We show here that this result also takes place if the monatomic gas* obeys quantum statistics (Fermi or Bose), provided that the energy is a power function of the momentum of the particle.

We write the kinetic equation for the distribution function n of the gas under consideration:

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}} \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}} \frac{\partial H}{\partial \mathbf{r}} = I(n). \tag{2}$$

Here $H(\mathbf{p}, \mathbf{r})$ is the Hamiltonian of the particle, $I(n)$ is the collision integral. Let there be macroscopic motion of the gas with velocity $\mathbf{u}$, for which $\operatorname{div} \mathbf{u} \neq 0$. In this case the unexcited distribution function is equal to

$$n = \frac{1}{\exp\{(\varepsilon - \mu - \mathbf{pu})/T\} + 1} \tag{3}$$

(the sign is negative in the case of Bose statistics, positive for Fermi statistics, $\epsilon(p)$ is the energy of the particle in the quiescent gas, μ is the chemical potential. In the absence of an external field the equality $H = \epsilon(p)$ holds. We substitute the n of Eq. (3) in the left side of the kinetic equation; for simplicity we shall consider the case here for which $\mathbf{u} = 0$ (but $\operatorname{div} \mathbf{u} \neq 0$). Moreover, inasmuch as we are interested only in the second viscosity, i.e., in terms in the momentum flux which are porportional to $\operatorname{div} \mathbf{u}$, it is reasonable that temperature gradients can be considered absent. For such a case we obtain for the left side of the kinetic equation

$$n'\left\{-\frac{\partial}{\partial t} \frac{\mu}{T} - \frac{\varepsilon}{T^2} \frac{\partial T}{\partial t} - \frac{1}{T} \frac{\partial \varepsilon}{\partial \mathbf{p}} \nabla (\mathbf{p}, \mathbf{u})\right\} = I(n). \tag{4}$$

We now consider the equation of continuity for the entropy σ and the density ρ at $\mathbf{u} = 0$, which have the following forms:

$$\frac{\partial \rho}{\partial t} + \rho \operatorname{div} \mathbf{u} = 0,\quad \frac{\partial \sigma}{\partial t} = 0 \tag{5}$$

(adiabatic condition).

In this case, Eq. (4) transforms to

$$n'\left\{\left[\rho \frac{\partial}{\partial \rho}\left(\frac{\mu}{T}\right)_\sigma + \frac{\varepsilon}{T^2}\left(\frac{\partial T}{\partial \rho}\right)_\sigma \rho - \frac{1}{3T} p \frac{\partial \varepsilon}{\partial p}\right] \operatorname{div} \mathbf{u} - \frac{1}{2T}\left(\frac{\sigma \varepsilon}{\partial p_i} p_k - \frac{1}{3} \delta_{ik} \frac{\partial \varepsilon}{\partial p} p\right) \times \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial u_l}{\partial x_l}\right)\right\} = I(n). \tag{6}$$

The second viscosity is absent in this case if the expression in square brackets on the left side of Eq. (6) vanishes. We show that this takes place if the energy ϵ is proportional to some power of the momentum

$$\varepsilon = ap^n. \tag{7}$$

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* By a monatomic gas we understand a set of particles each of which is characterized by three degrees of freedom (the three component vector momentum p).

The potential Ω is defined by the following relation[2]

$$\Omega = \mp kTV \int \frac{4\pi^2 p\,dp}{(2\pi\hbar)^3} \ln\left(1 \pm \exp\left\{\frac{\mu - \varepsilon}{kT}\right\}\right). \tag{8}$$

Substituting Eq. (7) for the energy in Eq. (8), and carrying out the substitution of variables $p \to p\,T^{1/n}$, we obtain

$$\Omega = -pV = VT^{1+3/n} f(\mu/T), \tag{9}$$

where f is some function of a single argument. We take advantage now of a thermodynamic identity for Ω and compute the entropy σ. Inasmuch as Ω is a homogeneous function of μ and T of order $1 + 3/n$, we have

$$\sigma = \frac{(\partial\Omega/\partial T)_{V,\mu}}{(\partial\Omega/\partial\mu)_{T,V}} = \varphi\left(\frac{\mu}{T}\right). \tag{10}$$

Thus, for adiabatic processes (σ = const) the relation μ/T is a constant quantity[2], i.e.,

$$\frac{\partial}{\partial\rho}\left(\frac{\mu}{T}\right)_\sigma = 0. \tag{11}$$

Furthermore, from the relation $N = -(\partial\Omega/\partial\mu)_{T,V}$ it follows that for adiabatic processes, $VT^{3/n}$ = const, and, consequently,

$$\left(\frac{\partial T}{\partial\rho}\right)_\sigma = \frac{n}{3}\frac{T}{\rho}. \tag{12}$$

By considering Eqs. (11) and (12), we can convince ourselves that the following expression holds:

$$\rho\frac{\partial}{\partial\rho}\left(\frac{\mu}{T}\right)_\sigma + \frac{1}{T}\left(\frac{\varepsilon}{T}\left(\frac{\partial T}{\partial\rho}\right)_\sigma - \frac{1}{3}\frac{\partial\varepsilon}{\partial p}p\right) = 0. \tag{13}$$

Thus, in the case under consideration, ($\epsilon = ap^n$) the second viscosity vanishes. Thus, for example, the second viscosity is equal to zero in a photon gas ($\epsilon = cp$), and also in a monatomic gas in the ultra-relativistic case. Evidently, the second viscosity will vanish in the liquid isotope of helium with mass 3 (He^3), which represents a set of Fermi particles. It is easy to see that condition (7) is necessary in order that the second viscosity equal zero. Actually, according to Eq. (6), if the second viscosity vanishes, then it is necessary that for all values of momenta, the following expression vanishes:

$$\varepsilon\left(\frac{\partial T}{\partial\rho}\right)_\sigma \frac{\rho}{T} - \frac{1}{3}p\frac{\partial\varepsilon}{\partial p} = 0, \tag{14}$$

or also,

$$\frac{1}{3}\frac{\partial\ln\varepsilon}{\partial\ln p} = \left(\frac{\partial\ln T}{\partial\ln\rho}\right)_\sigma. \tag{15}$$

Consequently, the energy is proportional to a power of the momentum, for which the power n is given by

$$n = 3(\partial\ln T/\partial\ln\rho)_\sigma. \tag{16}$$

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Translated by R. T. Beyer
185

The Surface Energy Associated with a Tangential Velocity Discontinuity in Helium II

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ONE of the most essential problems of the theory of superfluidity is, as we have already had occasion to point out[1,2], the question concerning the character of the tangential discontinuity in the velocity v_s of the superfluid component of helium, at the boundary between the fluid and a wall. The existence of such a discontinuity follows from the fact that the helium atoms adhere to the wall (a solid body), while at the same time, from the macroscopic viewpoint, i.e., in the immediate vicinity of the wall, the tangential component of v_s at the wall is not equal to zero[3]. The well-known proof[3] for the possibility of superfluidity consists in this case of the establishment of the conditions for stability of the discontinuity at the wall. The discontinuity and the superfluid flow are completely stable, provided the flow velocity $v < v_c$:

$$v_c = [\varepsilon(p)/p]_{\min}, \tag{1}$$

where $\epsilon(p)$ and p are the energy and the momentum of the "excitations" which may appear in the liquid (both quantities are measured in the coordinate system associated with the liquid). Within the framework of the microscopic representation, the thickness of the discontinuity is clearly not equal to zero, but is in order of magnitude equivalent to the atomic distance $a \sim N^{1/3} \simeq 3.5 \times 10^{-8}$ ($N = 2.2 \times 10^{22}$ is the concentration of atoms in liquid helium). Analogous discontinuities, in accordance with references 4 and 5, may also exist within the bulk helium II, in which case the situation is even simpler, since the question of the possible influence of the wall material upon the character of the discontinuity does not arise*. A certain surface energy σ must be associated with the velocity discontinuity, since this discontinuity represents a local disturbance of the superflow, requiring the expenditure of energy[5]. As it seems to us, such a conclusion is even more natural from the quantum viewpoint, inasmuch as the Ψ-function for the He atoms changes at the discontinuity within a distance $\sim a$; whence the average additional kinetic energy associated with unit surface area of the discontinuity**,

$$\sigma \sim \frac{\hbar^2 N}{2m_{\mathrm{He}}a^2}a \sim \frac{\hbar^2}{2m_{\mathrm{He}}a^4} \sim 5\times 10^{-2}\ \frac{\mathrm{erg}}{\mathrm{cm}^2} \qquad (2)$$

The expression (2) for σ was derived in reference 4 on dimensional grounds; in reference 5 there was obtained from similar considerations the formula

$$\sigma \sim \rho_s (kT_\lambda U^4/\rho)^{1/3}, \qquad (3)$$

in which ρ_s and ρ are, respectively, the density of the superfluid component of the helium and the total density of the helium, $T_\lambda = 2.19^0$, and U is the second sound velocity. Setting $\rho_s \sim \rho \sim 0.15$ and $U = 2 \times 10^3$ cm/sec, we obtain $\sigma \sim 5 \times 10^{-2}$; i.e., the approximations (2) and (3) are essentially in agreement, as was to be expected. The nature of the approximations is such that even a value of $\sigma \sim 5 \times 10^{-3}$ is compatible with them [this value is obtained from (2) when the thickness of the transition layer $\sim 10a \sim 3 \times 10^{-7}$ cm]. Calculation of the surface energy σ at the velocity discontinuities within the bulk helium II is essential to an understanding of the peculiarities arising for velocities greater than the critical velocity v_c, and, in particular, for rotation of helium in a beaker[4,5]. As was pointed out long ago by Landau, the formula $v_c \sim \sqrt{\sigma/\rho_s d}$, where d is the width of the slit (or capillary) through which helium II flows, may be derived on dimensional grounds.

If we consider Eq. (2), it becomes clear that the formula $v_c \sim (\hbar/m_{\mathrm{He}})\sqrt{1/ad}$ derived in reference 4 agrees in essence with the preceeding. It is not, perhaps, superfluous to point out that an analogous result is obtained by application of the fundamental criterion (1). We assume, actually, that within helium II there may form a region of volume V and surface area S, isolated from the remainder of the liquid. Then $p = Mu$ and $\epsilon(p) = \frac{1}{2}Mu^2 + \sigma S$, where $M = \rho_s V$ is the corresponding mass and u is the velocity of motion of the region in the coordinate system associated with the liquid. In this case, in accordance with Eq. (1)

$$v_c = \left[\frac{\frac{Mu^2}{2} + \sigma S}{Mu}\right]_{\min} = \sqrt{\frac{2\sigma}{\rho_s}\left(\frac{S}{V}\right)_{\min}} \qquad (4)$$

$$\sim \sqrt{\frac{\sigma}{\rho_s d}} \sim \frac{\hbar}{m_{\mathrm{He}}}\sqrt{\frac{1}{ad}} \sim \frac{0.1 \text{ to } 1}{\sqrt{d}}\ \frac{\mathrm{cm}}{\mathrm{sec}}$$

where d is the width of the slit or capillary, so that $(S/V)_{\min} \sim 1/d$. Here $u_{\min} \sim v_c$; i.e., the region formed (playing the role of the "elementary excitation" of reference 3) is at rest relative to the capillary walls. The minimal energy $\epsilon_{\min} = \frac{1}{2}Mu^2_{\min} + \sigma S = 2\sigma S$. The relation (4), within the limits of the very low accuracy achieved, agrees with experiment[7]; in any event, it agrees better than the relation $v_c \sim \hbar/m_{\mathrm{He}}d$ (cf. reference 2).

In view of all that has just been said, it now appears to us that the most natural explanation for the properties of the critical processes in helium is to be sought, neither through consideration of the quantum character of the excitations[2,7] nor through investigation of the surface excitations[2], but rather in the results of a study of the possibilities for formation of discontinuities[4,5].

The principle reason for the present letter, however, is the wish to emphasize another circumstance --- the necessity for calculating the surface energy σ' associated with the velocity discontinuity in the vicinity of the boundary between the helium II and a solid wall. According to all of the data, as has been said, such a discontinuity must necessarily exist, and therefore, the existence of a surface energy $\sigma' \sim \sigma$ is difficult to doubt. In order that the discontinuity should not leave the wall (which, apparently, is the case for $v < v_c$), it is necessary for the inequality $\sigma' < \sigma$ to be fulfilled (σ being the surface energy for a discontinuity within the bulk helium II)***. The value of σ' may to a certain extent depend upon the material of the wall, which is of interest from the standpoint of the possibility, considered below, of determining the influence of the surface energy σ' upon the flow of helium II. This influence should, first of all, manifest itself by the existence of a certain minimum energy $\sigma' S$ required for the setting in motion of a solid body of surface area S in helium II. It is obvious that a similar expenditure of energy should occur as well in the establishment of flow through slits and capillaries. Here, clearly, we are concerned with an effect analogous to that seen in the presence of so-called dry friction between solid surfaces. The situation is more complicated in the case of non-steady motion, since the disappearance of the surface energy σ' (the same applies as well to σ) cannot follow instantaneously upon the stopping of the body, and, apparently, metastable "discontinuities" can exist corresponding to zero velocity of relative motion between the wall and the helium II. The problem of the mechanism and the relaxation time τ, and also of the nature of such "discontinuities" (these appear to be strata in which the Ψ-function is perturbed relative to the corresponding lowest state), remains unclear Thus, as regards the influence of the energy σ' upon nonsteady flow, it is difficult to make any but purely qualitative statements. For example, in determining the viscosity of helium II from the damping of the oscillations of a disk, this damping may for $v < v_c$ be explained partially by the formation of discontinuities at the surface of the disk. Here, in the quasistationary case, an energy $4\sigma' S$ must be expended on the formation of discontinuities during the period θ of the oscillations, S being the surface area of the disk. Actually, however, in the experiments[8,9] the damping corresponding to $\sigma \sim 10^{-2}$ is 4 to 5 orders of magnitude smaller than the indicated quasistationary value, which demonstrates that the inequality $\tau \gg \theta \sim 10$ sec is fulfilled. Nevertheless, it is not impossible that the contribution to the damping associated with the formation of discontinuities is substantial, and that with it is to be connected the disparity between the results of the experiments[8,9] with oscillating disks and the measurements of the viscosity from the moment developed in rotating two coaxial cylinders relative to one another[10] (in the latter case the process is stationary, and the damping must be due solely to the viscosity); in reference 10 lower values are obtained for the damping at a low temperature, than in references 8 and 9, which accords with what has been said. The effect of the discontinuities may also be responsible for the peculiarities in the damping of a disk at large amplitudes[9]. In virtue of what has been stated, it seems to us that discontinuites in the flow velocity of helium II in the vicinity of walls deserve close attention.

The author is obliged to Academician L. D. Landau and to Professor E. M. Lifshitz for their discussion of this problem.

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* At the same time, the hypothesis of the possible existence of velocity discontinuities within the bulk helium II becomes especially likely when it is considered that such discontinuities are known to exist in the vicinity of the wall.

** We note that in the theory of superconductivity[6] the surface energy σ_{ns} at the boundary between the superconducting and normal phases can be successfully evaluated from similar considerations. Thus, assuming that the thickness of the transition layer between the phases δ_0/κ (cf reference 6), we obtain

$$\sigma_{ns} \sim \frac{\hbar^2 n_s (\delta_0/\varkappa)}{2m(\delta_0/\varkappa)^2} \sim \frac{H_{km}^2 \delta_0}{2\pi\varkappa},$$

which agrees with more exact calculations[6] (for the symbols, reference 6, noting that $n_s = mc^2/4\pi e^2 \delta_0^2$ and $\kappa^2 = (2e^2/\hbar^2 c^2) H_{km}^2 \delta_0^4$).

*** If σ' and σ depend on v_s [for example, if $\sigma' = \sigma'(v_s \to 0) + bv_s^2$], then, in principle, it is possible for critical processes to develop, in conjunction with the fact that for $v \geq v_c$, $\sigma' \geq \sigma$. We note further that if we set $\sigma = \propto u^2$ (cf. reference 4), then $v_c = 0$. This circumstance, together with a number of others, indicates that (provided that all of the ideas under consideration are correct) the surface energies σ and σ' tend to a limit different from zero as $v_s \to 0$.

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Translated by S. D. Elliott
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The Effective Density of Rotating Liquid Helium II

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LANDAU and Lifshitz have shown[1] that in the rotation of a vessel containing He II, the normal part of the helium rotates as a whole, while

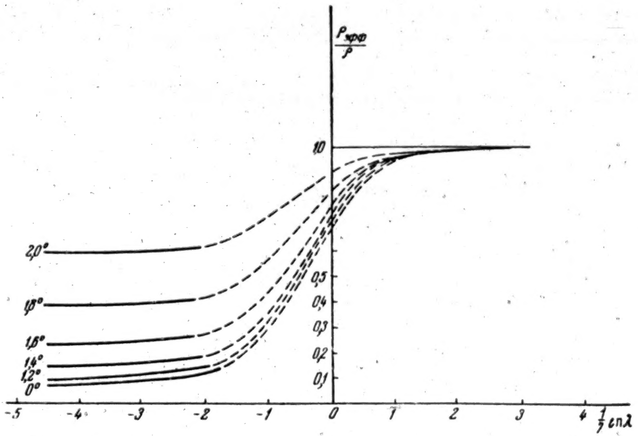

FIG. 1. Dependence of ρ_{eff}/ρ on the angular velocity Ω for various temperatures (the numbers on the curves are in degrees K).

for the superfluid, the cylindrical volume is divided into a series of coaxial cylindrical layers, in each of which superfluid motion takes place with a velocity distribution according to the law

$$v_s^{(i)} = \frac{b_i}{r}\ ;\ b_i = \frac{\Omega}{2}\,\frac{r_i^2 - r_{i+1}^2}{\ln(r_i/r_{i+1})} \tag{1}$$

We make use of the notation of the research just mentioned. The values of the radius of the boundaries of a section r_i are defined in reference 1 for two limiting cases:

a) slow rotation ($\lambda = \rho_s\,\Omega^2 R^3/8\alpha \ll 1$):

$$\sqrt{\xi_{i+1}}\ln\frac{1}{\sqrt{\xi_{i+1}}} = \frac{1}{2}\sqrt{\lambda x_i^3}, \tag{2}$$

$$\xi_{i+1} = \frac{x_{i+1}}{x_i},\quad x_i = \frac{r_i}{R}\ ;$$

b) fast rotation ($\lambda \gg 1$):

$$\Delta = \frac{r_i - r_{i+1}}{R} = \left(\frac{3}{8\lambda}\right)^{1/3}. \tag{3}$$

The latter formula is correct for layers located not too close to the axis of the cylinder.

The fact that the superfluid component takes part in the rotation leads to the unusual dependence of the momentum of the helium on the angular velocity (we recall that it was first believed that only the normal component rotated). This dependence is obtained by substituting, in the expression for the momentum per unit length of the vessel containing the helium,

$$M = 2\pi\int_0^R (\rho_n v_n + \rho_s v_s)\,r^2\,dr \tag{4}$$

the values $v_n = \Omega r$ and v_s in accord with Eqs. (1):

$$M = \frac{\pi\Omega R^4}{2}\left\{\rho_n + \rho_s\sum_j \frac{(x_j^2 - x_{j+1}^2)^2}{\ln(x_j/x_{j+1})}\right\}. \tag{5}$$

If we make use of the latter expression, it is natural to define the effective density of the rotating helium in the following manner:

$$M = \frac{\pi\Omega R^4}{2}\,\rho_{\mathrm{eff}},\ \rho_{\mathrm{eff}} = \rho_n + \rho_s\sum_j\frac{(x_j^2 - x_{j+1}^2)^2}{\ln(x_j/x_{j+1})}. \tag{6}$$

In the limiting cases, we have

a) slow rotation [in accord with Eq. (2)]:

$$\rho_{\mathrm{eff}} = \rho_n + \rho_s\frac{1}{\ln(1/x_1)}, \tag{6'}$$

$$\sqrt{x_1}\ln\frac{1}{\sqrt{x_1}} = \frac{\sqrt{\lambda}}{2}\quad (\lambda \ll 1);$$

b) fast rotation [in accord with Eq. (2)]:

$$\rho_{\mathrm{eff}} = \rho_n + \rho_s\left[1 - \frac{1}{3}\left(\frac{3}{8\lambda}\right)^{1/3}\right]\quad (\lambda \gg 1). \tag{6''}$$

The dependence of the isotherms ρ_{eff}/ρ on the angular velocity is shown in the Figure. The values of ρ_n/ρ are taken from the review of Lifshitz[2] (Fig. 210, p. 404).

Experimental investigation of the dependence of $\rho_{\mathrm{eff}}(\Omega)$ provides, it seems to us, a definite possibility of finding the stratification of He II predicted by Landau and Lifshitz. In this case, in order that the meniscus effect does not distort the results, it is necessary that the liquid fill the vessel completely.

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[1] L. D. Landau and E. M. Lifshitz, Dokl. Akad. Nauk SSSR 100, 669 (1955)

[2] E. M. Lifshitz, *Superfluidity (Theory)*. (Appendix to the monograph of V. Keesom, *Helium*, 1949)

Translated by R. T. Beyer
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ERRATUM

Khartman, Leont'eva, Siniaviskii and Vasil'ev, Soviet Phys. 1, 537-545 (1955). The two halves of the circuit diagram have been transposed. The half that appears on page 541 should be at the left; the half on page 540 should be at the right.

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